The Rectangular Coordinate System and Circles

Each of the pairs of numbers $(1, 2)$, $(-1, 5)$, and $(3, 7)$ is an example of an ordered pair; that is, a pair of numbers written within parentheses in which the order of the numbers is important. The two numbers are the components of the ordered pair. An ordered pair is graphed using two number lines that intersect at right angles at the zero points, as shown in Figure 1. The common zero point is called the origin. The horizontal line, the $x$-axis, represents the first number in an ordered pair, and the vertical line, the $y$-axis, represents the second. The $x$-axis and the $y$-axis make up a rectangular (or Cartesian) coordinate system. The axes form four quadrants, numbered I, II, III, and IV as shown in Figure 2. (A point on an axis is not considered to be in any of the four quadrants.)

We locate, or plot, the point on the graph that corresponds to the ordered pair $(3, 1)$ by going three units from zero to the right along the $x$-axis, and then one unit up parallel to the $y$-axis. The point corresponding to the ordered pair $(3, 1)$ is labeled A in Figure 2. The phrase “the point corresponding to the ordered pair $(3, 1)$” often is abbreviated “the point $(3, 1)$.” The numbers in an ordered pair are called the coordinates of the corresponding point.

The parentheses used to represent an ordered pair also are used to represent an open interval (introduced in an earlier chapter). In general, there should be no confusion between these symbols because the context of the discussion tells us whether we are discussing ordered pairs or open intervals.

Distance Formula

Suppose that we wish to find the distance between two points, say $(3, -4)$ and $(-5, 3)$. The Pythagorean theorem allows us to do this. In Figure 3 on the next page, we see that the vertical line through $(-5, 3)$ and the horizontal line through $(3, -4)$ intersect at the point $(-5, -4)$. Thus, the point $(-5, -4)$ becomes the vertex of the right angle in a right triangle. By the Pythagorean Theorem, the square of the length of the hypotenuse, $d$, of the right triangle in Figure 3 is equal to the sum of the squares of the lengths of the two legs $a$ and $b$:

$$d^2 = a^2 + b^2.$$  

The length $a$ is the distance between the endpoints of that leg. Since the $x$-coordinate of both points is $-5$, the side is vertical, and we can find $a$ by finding the difference between the $y$-coordinates. Subtract $-4$ from $3$ to get a positive value of $a$.

$$a = 3 - (-4) = 7$$


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**Double Descartes** After the French postal service issued the above stamp in honor of René Descartes, sharp eyes noticed that the title of Descartes’s most famous book was wrong. Thus a second stamp (see facing page) was issued with the correct title. The book in question, *Discourse on Method*, appeared in 1637. In it Descartes rejected traditional Aristotelian philosophy, outlining a universal system of knowledge that was to have the certainty of mathematics. For Descartes, method was analysis, going from self-evident truths step-by-step to more distant and more general truths. One of these truths is his famous statement, “I think, therefore I am.” (Thomas Jefferson, also a rationalist, began the Declaration with the words, “We hold these truths to be self-evident.”)
Similarly, find $b$ by subtracting $-5$ from $3$.

\[ b = 3 - (-5) = 8 \]

Substituting these values into the formula, we have

\[ d^2 = a^2 + b^2 \]
\[ d^2 = 7^2 + 8^2 \quad \text{Let } a = 7 \text{ and } b = 8. \]
\[ d^2 = 49 + 64 \]
\[ d^2 = 113 \]
\[ d = \sqrt{113}. \]

Therefore, the distance between $(−5, 3)$ and $(3, −4)$ is $\sqrt{113}$.

**Distance Formula**

The distance between the points $(x_1, y_1)$ and $(x_2, y_2)$ is

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \]

This result is called the **distance formula**.

The small numbers 1 and 2 in the ordered pairs $(x_1, y_1)$ and $(x_2, y_2)$ are called subscripts. We read $x_1$ as “$x$ sub 1.” Subscripts are used to distinguish between different values of a variable that have a common property. For example, in the ordered pairs $(−3, 5)$ and $(6, 4)$, $−3$ can be designated as $x_1$ and $6$ as $x_2$. Their common property is that they are both $x$ components of ordered pairs. This idea is used in the following example.
EXAMPLE 1 Find the distance between \((-3, 5)\) and \((6, 4)\).

When using the distance formula to find the distance between two points, designating the points as \((x_1, y_1)\) and \((x_2, y_2)\) is arbitrary. Let us choose \((x_1, y_1) = (-3, 5)\) and \((x_2, y_2) = (6, 4)\).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\\
= \sqrt{(6 - (-3))^2 + (4 - 5)^2}\\
= \sqrt{9^2 + (-1)^2}\\
= \sqrt{82}.
\]

Midpoint Formula The midpoint of a line segment is the point on the segment that is equidistant from both endpoints. Given the coordinates of the two endpoints of a line segment, it is not difficult to find the coordinates of the midpoint of the segment.

**Midpoint Formula** The coordinates of the midpoint of the segment with endpoints \((x_1, y_1)\) and \((x_2, y_2)\) are

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).
\]

In words, the coordinates of the midpoint of a line segment are found by calculating the averages of the \(x\)- and \(y\)-coordinates of the endpoints.

EXAMPLE 2 Find the coordinates of the midpoint of the line segment with endpoints \((8, -4)\) and \((-9, 6)\).

Using the midpoint formula, we find that the coordinates of the midpoint are

\[
\left( \frac{8 + (-9)}{2}, \frac{-4 + 6}{2} \right) = \left( -\frac{1}{2}, 1 \right).
\]

EXAMPLE 3 Figure 5 on the next page depicts how the purchasing power of the dollar declined from 1990 to 2000, based on changes in the Consumer Price Index. In this model, the base period is 1982–1984. For example, it would have cost $1.00 to purchase in 1990 what $.766 would have purchased during the base period. Use the graph to estimate the purchasing power of the dollar in 1995, and compare it to the actual figure of $.656.

The year 1995 lies halfway between 1990 and 2000, so we must find the coordinates of the midpoint of the segment that has endpoints \((1990, .766)\) and \((2000, .581)\). This is given by

\[
\left( \frac{1990 + 2000}{2}, \frac{.766 + .581}{2} \right) = (1995, .6735).
\]
Thus, based on this procedure, the purchasing power of the dollar to the nearest thousandth was $0.674 in 1995. This is fairly close to the actual figure of $0.656.

**FIGURE 5**

**Circles** An application of the distance formula leads to one of the most familiar shapes in geometry, the circle. A circle is the set of all points in a plane that lie a fixed distance from a fixed point. The fixed point is called the center and the fixed distance is called the radius.

**EXAMPLE 4** Find an equation of the circle with radius 3 and center at (0, 0), and graph the circle.

If the point \((x, y)\) is on the circle, the distance from \((x, y)\) to the center \((0, 0)\) is 3, as shown in Figure 6. By the distance formula,

\[
\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d
\]

\[
\sqrt{(x - 0)^2 + (y - 0)^2} = 3
\]

\[x_1 = 0, y_1 = 0, x_2 = x, y_2 = y\]

\[x^2 + y^2 = 9.\] Square both sides.

An equation of this circle is \(x^2 + y^2 = 9\). It can be graphed by locating all points three units from the origin.

**FIGURE 6**
EXAMPLE 5  Find an equation for the circle that has its center at \( (4, -3) \) and radius 5, and graph the circle.

Again use the distance formula.

\[
\sqrt{(x - 4)^2 + (y + 3)^2} = 5
\]

\[
(x - 4)^2 + (y + 3)^2 = 25
\]


The graph of this circle is shown in Figure 7.

Examples 4 and 5 can be generalized to get an equation of a circle with radius \( r \) and center at \( (h, k) \). If \( (x, y) \) is a point on the circle, the distance from the center \( (h, k) \) to the point \( (x, y) \) is \( r \). Then by the distance formula, \( \sqrt{(x - h)^2 + (y - k)^2} = r \). Squaring both sides gives the following equation of a circle.

**Equation of a Circle**

The equation of a circle of radius \( r \) with center at \( (h, k) \) is

\[
(x - h)^2 + (y - k)^2 = r^2.
\]

In particular, a circle of radius \( r \) with center at the origin has equation

\[
x^2 + y^2 = r^2.
\]

EXAMPLE 6  Find an equation of the circle with center at \( (-1, 2) \) and radius 4.

Let \( h = -1, k = 2, \) and \( r = 4 \) in the general equation above to get

\[
(x - (-1))^2 + (y - 2)^2 = 4 \quad \text{and} \quad (x + 1)^2 + (y - 2)^2 = 16.
\]

In the equation found in Example 5, multiplying out \( (x - 4)^2 \) and \( (y + 3)^2 \) and then combining like terms gives

\[
(x - 4)^2 + (y + 3)^2 = 25
\]

\[
x^2 - 8x + 16 + y^2 + 6y + 9 = 25
\]

\[
x^2 + y^2 - 8x + 6y = 0.
\]

This result suggests that an equation that has both \( x^2 \) and \( y^2 \) terms may represent a circle. The next example shows how to tell, using the method of completing the square.

EXAMPLE 7  Graph \( x^2 + y^2 + 2x + 6y - 15 = 0 \).

Since the equation has \( x^2 \) and \( y^2 \) terms with equal coefficients, its graph might be that of a circle. To find the center and radius, complete the squares on \( x \) and \( y \) as follows. (See the previous chapter, where completing the square is introduced.)

\[
x^2 + y^2 + 2x + 6y = 15 \quad \text{Add 15 to both sides.}
\]

\[
(x^2 + 2x\phantom{y}) + (y^2 + 6y\phantom{x}) = 15 \quad \text{Rewrite in anticipation of completing the square.}
\]

\[
(x^2 + 2x + 1) + (y^2 + 6y + 9) = 15 + 1 + 9 \quad \text{Complete the squares on both} \ x \text{and} \ y.
\]

\[
(x + 1)^2 + (y + 3)^2 = 25 \quad \text{Factor on the left and add on the right.}
\]

The final equation shows that the graph is a circle with center at \( (-1, -3) \) and radius 5. The graph is shown in Figure 8.
The final example in this section shows how equations of circles can be used in locating the epicenter of an earthquake.

**Example 8** Seismologists can locate the epicenter of an earthquake by determining the intersection of three circles. The radii of these circles represent the distances from the epicenter to each of three receiving stations. The centers of the circles represent the receiving stations.

Suppose receiving stations $A$, $B$, and $C$ are located on a coordinate plane at the points $(1, 4)$, $(-3, -1)$, and $(5, 2)$. Let the distances from the earthquake epicenter to the stations be 2 units, 5 units, and 4 units, respectively. See Figure 9. Where on the coordinate plane is the epicenter located?

Graphically, it appears that the epicenter is located at $(1, 2)$. To check this algebraically, determine the equation for each circle and substitute $x = 1$ and $y = 2$.

**Station A:**

\[
(x - 1)^2 + (y - 4)^2 = 4 \quad \quad (x + 3)^2 + (y + 1)^2 = 25
\]

\[
(x - 1)^2 + (2 - 4)^2 = 4 \quad \quad (1 + 3)^2 + (2 + 1)^2 = 25
\]

\[
0 + 4 = 4 \quad \quad 16 + 9 = 25
\]

\[
4 = 4 \quad \quad 25 = 25
\]

**Station B:**

\[
(x - 5)^2 + (y - 2)^2 = 16
\]

\[
(1 - 5)^2 + (2 - 2)^2 = 16
\]

\[
16 + 0 = 16
\]

\[
16 = 16
\]

Thus, we can be sure that the epicenter lies at $(1, 2)$.