Use the method of Example 8 or your own method to solve each problem.

69. **Travel Times of Trains**  A train leaves Little Rock, Arkansas, and travels north at 85 kilometers per hour. Another train leaves at the same time and travels south at 95 kilometers per hour. How long will it take before they are 315 kilometers apart?

<table>
<thead>
<tr>
<th>Rate</th>
<th>Time</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>First train</td>
<td>85</td>
<td>$t$</td>
</tr>
<tr>
<td>Second train</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

70. **Travel Times of Steamers**  Two steamers leave a port on a river at the same time, traveling in opposite directions. Each is traveling 22 miles per hour. How long will it take for them to be 110 miles apart?

<table>
<thead>
<tr>
<th>Rate</th>
<th>Time</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>First steamer</td>
<td>$t$</td>
<td></td>
</tr>
<tr>
<td>Second steamer</td>
<td>22</td>
<td></td>
</tr>
</tbody>
</table>

71. **Travel Times of Commuters**  Nancy and Mark commute to work, traveling in opposite directions. Nancy leaves the house at 8:00 A.M. and averages 35 miles per hour. Mark leaves at 8:15 A.M. and averages 40 miles per hour. At what time will they be 140 miles apart?

72. **Travel Times of Bicyclers**  Jeff leaves his house on his bicycle at 8:30 A.M. and averages 5 miles per hour. His wife, Joan, leaves at 9:00 A.M., following the same path and averaging 8 miles per hour. At what time will Joan catch up with Jeff?

73. **Distance Traveled to Work**  Tri drives his car to work, the trip takes 30 minutes. When he rides the bus, it takes 45 minutes. The average speed of the bus is 12 miles per hour less than his speed when driving. Find the distance he travels to work.

74. **Distance Traveled to School**  Latoya can get to school in 15 minutes if she rides her bike. It takes her 45 minutes if she walks. Her speed when walking is 10 miles per hour slower than her speed when riding. How far does she travel to school?

75. **Time Traveled by a Pleasure Boat**  A pleasure boat on the Mississippi River traveled from Baton Rouge to New Orleans with a stop at White Castle. On the first part of the trip, the boat traveled at an average speed of 10 miles per hour. From White Castle to New Orleans the average speed was 15 miles per hour. The entire trip covered 100 miles. How long did the entire trip take if the two parts each took the same number of hours?

76. **Time Traveled on a Visit**  Steve leaves Nashville to visit his cousin David in Napa, 80 miles away. He travels at an average speed of 50 miles per hour. One-half hour later David leaves to visit Steve, traveling at an average speed of 60 miles per hour. How long after David leaves will they meet?

### 7.3 Ratio, Proportion, and Variation

**Ratio**  One of the most frequently used mathematical concepts in everyday life is ratio. A baseball player’s batting average is actually a ratio. The slope, or pitch, of a roof on a building may be expressed as a ratio. Ratios provide a way of comparing two numbers or quantities.

**Ratio**

A ratio is a quotient of two quantities. The ratio of the number $a$ to the number $b$ is written

$$a \text{ to } b, \quad \frac{a}{b}, \quad \text{or} \quad a:b.$$
CHAPTER 7
The Basic Concepts of Algebra

EXAMPLE 1 Write a ratio for each word phrase.

(a) the ratio of 5 hours to 3 hours

This ratio can be written as $\frac{5}{3}$.

(b) the ratio of 5 hours to 3 days

First convert 3 days to hours: $3 \text{ days} = 3 \cdot 24 = 72 \text{ hours}$. The ratio of 5 hours to 3 days is thus $\frac{5}{72}$.

Proportion We now define a special type of equation called a proportion.

A proportion is a statement that says that two ratios are equal.

For example,

$$\frac{3}{4} = \frac{15}{20}$$

is a proportion that says that the ratios $\frac{3}{4}$ and $\frac{15}{20}$ are equal. In the proportion

$$\frac{a}{b} = \frac{c}{d},$$

$a$, $b$, $c$, and $d$ are the terms of the proportion. The $a$ and $d$ terms are called the extremes, and the $b$ and $c$ terms are called the means. We can read the proportion $\frac{a}{b} = \frac{c}{d}$ as “$a$ is to $b$ as $c$ is to $d$.” Beginning with this proportion and multiplying both sides by the common denominator, $bd$, gives

$$bd \cdot \frac{a}{b} = bd \cdot \frac{c}{d}$$

That is, the product of the extremes equals the product of the means. The products $ad$ and $bc$ can also be found by multiplying diagonally.

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{bc}{ad}$$

This is called cross multiplication and $ad$ and $bc$ are called cross products.

Cross Products

If $\frac{a}{b} = \frac{c}{d}$, then the cross products $ad$ and $bc$ are equal.

Also, if $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$ (as long as $b \neq 0$, $d \neq 0$).
From the rule given on page 340, if \( \frac{a}{b} = \frac{c}{d} \) then \( ad = bc \). However, if \( \frac{a}{c} = \frac{b}{d} \), then \( ad = cb \), or \( ad = bc \). This means that the two proportions are equivalent and

\[
\frac{a}{b} = \frac{c}{d} \text{ can also be written as } \frac{a}{c} = \frac{b}{d}.
\]

Sometimes one form is more convenient to work with than the other.

Four numbers are used in a proportion. If any three of these numbers are known, the fourth can be found.

**Example 2**  Solve each proportion.

(a) \( \frac{63}{x} = \frac{9}{5} \)

The cross products must be equal.

\[
63 \cdot 5 = 9x \quad \text{Cross products}
\]

\[
315 = 9x \\
35 = x \quad \text{Divide by 9.}
\]

The solution set is \( \{35\} \).

(b) \( \frac{8}{5} = \frac{12}{r} \)

\[
8r = 5 \cdot 12 \quad \text{Set the cross products equal.}
\]

\[
8r = 60 \\
r = \frac{60}{8} = \frac{15}{2} \quad \text{Divide by 8; express in lowest terms.}
\]

The solution set is \( \left\{\frac{15}{2}\right\} \).

**Example 3**  Solve the equation

\[
\frac{m - 2}{5} = \frac{m + 1}{3}.
\]

Find the cross products, and set them equal to each other.

\[
3(m - 2) = 5(m + 1) \quad \text{Be sure to use parentheses.}
\]

\[
3m - 6 = 5m + 5 \quad \text{Distributive property}
\]

\[
3m = 5m + 11 \quad \text{Add 6.}
\]

\[
-2m = 11 \quad \text{Subtract 5m.}
\]

\[
m = -\frac{11}{2} \quad \text{Divide by -2.}
\]

The solution set is \( \left\{-\frac{11}{2}\right\} \).
While the cross product method is useful in solving equations of the types found in Examples 2 and 3, it cannot be used directly if there is more than one term on either side. For example, you cannot use the method directly to solve the equation

\[
\frac{4}{x} + 3 = \frac{1}{9},
\]

because there are two terms on the left side.

**Example 4**  
Biologists use algebra to estimate the number of fish in a lake. They first catch a sample of fish and mark each specimen with a harmless tag. Some weeks later, they catch a similar sample of fish from the same areas of the lake and determine the proportion of previously tagged fish in the new sample. The total fish population is estimated by assuming that the proportion of tagged fish in the new sample is the same as the proportion of tagged fish in the entire lake.

Suppose biologists tag 300 fish on May 1. When they return on June 1 and take a new sample of 400 fish, 5 of the 400 were previously tagged. Estimate the number of fish in the lake.

Let \( x \) represent the number of fish in the lake. Set up and solve a proportion.

\[
\frac{300}{x} = \frac{5}{400}
\]

\[
x = 24,000
\]

There are approximately 24,000 fish in the lake.

**Unit pricing** — deciding which size of an item offered in different sizes produces the best price per unit — uses proportions. Suppose you buy 36 ounces of pancake syrup for $3.89. To find the price per unit, set up and solve a proportion.

\[
\frac{36 \text{ ounces}}{1 \text{ ounce}} = \frac{3.89}{x}
\]

\[
36x = 3.89
\]

\[
x = \frac{3.89}{36}
\]

\[x \approx .108\] Use a calculator.

Thus, the price for 1 ounce is $.108, or about 11 cents. Notice that the unit price is the ratio of the cost for 36 ounces, $3.89, to the number of ounces, 36, which means that the unit price for an item is found by dividing the cost by the number of units.

<table>
<thead>
<tr>
<th>Size</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>36-ounce</td>
<td>$3.89</td>
</tr>
<tr>
<td>24-ounce</td>
<td>$2.79</td>
</tr>
<tr>
<td>12-ounce</td>
<td>$1.89</td>
</tr>
</tbody>
</table>

**Example 5** Besides the 36-ounce size discussed earlier, the local supermarket carries two other sizes of a popular brand of pancake syrup, priced as shown in the table. Which size is the best buy? That is, which size has the lowest unit price?

To find the best buy, divide the price by the number of units to get the price per ounce. Each result in the table on the next page was found by using a calculator and rounding the answer to three decimal places.
Since the 36-ounce size produces the lowest price per unit, it would be the best buy. (Be careful: Sometimes the largest container does not produce the lowest price per unit.)

**Variation** Suppose that a carpet cleaning service charges $22.50 per room to shampoo a carpet. The table in the margin shows the relationship between the number of rooms cleaned and the cost of the total job for 1 through 5 rooms.

If we divide the cost of the job by the number of rooms, in each case we obtain the quotient, or ratio, 22.50 (dollars per room). Suppose that we let \( x \) represent the number of rooms and \( y \) represent the cost for cleaning that number of rooms. Then the relationship between \( x \) and \( y \) is given by the equation

\[
\frac{y}{x} = 22.50
\]

or

\[
y = 22.50x.
\]

This relationship between \( x \) and \( y \) is an example of direct variation.

### Direct Variation

\( y \) **varies directly as** \( x \), or \( y \) **is directly proportional to** \( x \), if there exists a nonzero constant \( k \) such that

\[
y = kx
\]

or, equivalently,

\[
\frac{y}{x} = k.
\]

The constant \( k \) is a numerical value called the **constant of variation**.

### Example 6

Suppose \( y \) varies directly as \( x \), and \( y = 50 \) when \( x = 20 \). Find \( y \) when \( x = 14 \).

Since \( y \) varies directly as \( x \), there exists a constant \( k \) such that \( y = kx \). Find \( k \) by replacing \( y \) with 50 and \( x \) with 20.

\[
\begin{align*}
y &= kx \\
50 &= k \cdot 20 \\
\frac{5}{2} &= k
\end{align*}
\]
Since \( y = kx \) and \( k = \frac{5}{2} /H11080 \),

\[
y = \frac{5}{2} x.
\]

Now find \( y \) when \( x = 14 /H33527 \).

\[
y = \frac{5}{2} \cdot 14 = 35
\]

The value of \( y \) is 35 when \( x = 14 /H11080 \).

**EXAMPLE 7**  Hooke’s law for an elastic spring states that the distance a spring stretches is directly proportional to the force applied. If a force of 150 pounds stretches a certain spring 8 centimeters, how much will a force of 400 pounds stretch the spring? See Figure 5.

If \( d \) is the distance the spring stretches and \( f \) is the force applied, then \( d = kf \) for some constant \( k \). Since a force of 150 pounds stretches the spring 8 centimeters,

\[
d = kf \quad \text{Formula}
\]

\[
8 = k \cdot 150 \quad d = 8, f = 150
\]

\[
k = \frac{8}{150} = \frac{4}{75} \quad \text{Find } k.
\]

and \( d = \frac{4}{75} f \). For a force of 400 pounds,

\[
d = \frac{4}{75} (400) = \frac{64}{3}.
\]

Let \( f = 400 \).

The spring will stretch \( 64/3 \) centimeters if a force of 400 pounds is applied.

In summary, follow these steps to solve a variation problem.

**Solving a Variation Problem**

**Step 1:** Write the variation equation.

**Step 2:** Substitute the initial values and solve for \( k \).

**Step 3:** Rewrite the variation equation with the value of \( k \) from Step 2.

**Step 4:** Substitute the remaining values, solve for the unknown, and find the required answer.

In some cases one quantity will vary directly as a *power* of another.

**Direct Variation as a Power**

\( y \) varies directly as the \( n \)th power of \( x \) if there exists a real number \( k \) such that

\[
y = kx^n.
\]
An example of direct variation as a power involves the area of a circle. The formula for the area of a circle is

$$A = \pi r^2.$$ 

Here, $\pi$ is the constant of variation, and the area varies directly as the square of the radius.

**EXAMPLE 8** The distance a body falls from rest varies directly as the square of the time it falls (here we disregard air resistance). If a skydiver falls 64 feet in 2 seconds, how far will she fall in 8 seconds?

**Step 1:** If $d$ represents the distance the skydiver falls and $t$ the time it takes to fall, then $d$ is a function of $t$, and

$$d = kt^2$$

for some constant $k$.

**Step 2:** To find the value of $k$, use the fact that the object falls 64 feet in 2 seconds.

Formula

$$d = kt^2$$

Let $d = 64$ and $t = 2$. 

$$k = 16$$

Find $k$.

**Step 3:** With this result, the variation equation becomes

$$d = 16t^2.$$ 

**Step 4:** Now let $t = 8$ to find the number of feet the skydiver will fall in 8 seconds.

Let $t = 8$.

The skydiver will fall 1024 feet in 8 seconds.

In direct variation where $k > 0$, as $x$ increases, $y$ increases, and similarly as $x$ decreases, $y$ decreases. Another type of variation is **inverse variation**.

**Inverse Variation**

$y$ varies inversely as $x$ if there exists a real number $k$ such that

$$y = \frac{k}{x},$$

or, equivalently,

$$xy = k.$$ 

Also, $y$ varies inversely as the $n$th power of $x$ if there exists a real number $k$ such that

$$y = \frac{k}{x^n}.$$
**EXAMPLE 9** The weight of an object above the earth varies inversely as the square of its distance from the center of Earth. A space vehicle in an elliptical orbit has a maximum distance from the center of Earth (apogee) of 6700 miles. Its minimum distance from the center of Earth (perigee) is 4090 miles. See Figure 6 (not to scale). If an astronaut in the vehicle weighs 57 pounds at its apogee, what does the astronaut weigh at the perigee?

![Figure 6](image)

If \( w \) is the weight and \( d \) is the distance from the center of Earth, then

\[
w = \frac{k}{d^2}
\]

for some constant \( k \). At the apogee the astronaut weighs 57 pounds and the distance from the center of Earth is 6700 miles. Use these values to find \( k \).

\[
57 = \frac{k}{(6700)^2}
\]

Let \( w = 57 \) and \( d = 6700 \).

\[
k = 57(6700)^2
\]

Then the weight at the perigee with \( d = 4090 \) miles is

\[
w = \frac{57(6700)^2}{(4090)^2} = 153 \text{ pounds.}
\]

Use a calculator.

It is common for one variable to depend on several others. For example, if one variable varies as the product of several other variables (perhaps raised to powers), the first variable is said to **vary jointly** as the others.

**EXAMPLE 10** The strength of a rectangular beam varies jointly as its width and the square of its depth. If the strength of a beam 2 inches wide by 10 inches deep is 1000 pounds per square inch, what is the strength of a beam 4 inches wide and 8 inches deep?

If \( S \) represents the strength, \( w \) the width, and \( d \) the depth, then

\[
S = kwd^2
\]

for some constant \( k \). Since \( S = 1000 \) if \( w = 2 \) and \( d = 10 \),

\[
1000 = k(2)(10)^2.
\]

Let \( S = 1000, w = 2, \) and \( d = 10 \).
Solving this equation for \( k \) gives
\[
1000 = k \cdot 2 \cdot 100 \\
1000 = 200k \\
k = 5 ,
\]
so
\[
S = 5wd^2 .
\]

Find \( S \) when \( w = 4 \) and \( d = 8 \) by substitution in \( S = 5wd^2 \).
\[
S = 5(4)(8)^2 = 1280 \quad \text{Let } w = 4 \text{ and } d = 8.
\]
The strength of the beam is 1280 pounds per square inch.

There are many combinations of direct and inverse variation. The final example shows a typical combined variation problem.

**EXAMPLE 11** The maximum load that a cylindrical column with a circular cross section can hold varies directly as the fourth power of the diameter of the cross section and inversely as the square of the height. A 9-meter column 1 meter in diameter will support 8 metric tons. See Figure 7. How many metric tons can be supported by a column 12 meters high and 2/3 meter in diameter?

Let \( L \) represent the load, \( d \) the diameter, and \( h \) the height. Then
\[
L = \frac{kd^4}{h^2} .
\]

Load varies directly as the 4th power of the diameter.

Load varies inversely as the square of the height.

Now find \( k \). Let \( h = 9, d = 1, \) and \( L = 8 \).
\[
8 = \frac{k(1)^4}{9^2} \quad h = 9, d = 1, L = 8
\]
\[
8 = \frac{k}{81}
\]
\[
k = 648
\]
Substitute 648 for \( k \) in the first equation.
\[
L = \frac{648d^4}{h^2}
\]

Now find \( L \) when \( h = 12 \) and \( d = 2/3 \).
\[
L = \frac{648\left(\frac{2}{3}\right)^4}{12^2} = \frac{648\left(\frac{16}{81}\right)}{144} = 648 \cdot \frac{16}{81} \cdot \frac{1}{144} = \frac{8}{9} \quad \text{Let } h = 12, d = \frac{2}{3}.
\]
The maximum load is 8/9 metric ton.