### Activity Symbol per Hour

<table>
<thead>
<tr>
<th>Activity</th>
<th>Symbol</th>
<th>per Hour</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volleyball</td>
<td>v</td>
<td>160</td>
</tr>
<tr>
<td>Golf</td>
<td>g</td>
<td>260</td>
</tr>
<tr>
<td>Canoeing</td>
<td>c</td>
<td>340</td>
</tr>
<tr>
<td>Swimming</td>
<td>s</td>
<td>410</td>
</tr>
<tr>
<td>Running</td>
<td>r</td>
<td>680</td>
</tr>
</tbody>
</table>

**Venn Diagrams and Subsets**

In every problem there is either a stated or implied universe of discourse. The universe of discourse includes all things under discussion at a given time. For example, in studying reactions to a proposal that a certain campus raise the minimum age of individuals to whom beer may be sold, the universe of discourse might be all the students at the school, the nearby members of the public, the board of trustees of the school, or perhaps all these groups of people.

In the mathematical theory of sets, the universe of discourse is called the universal set. The letter $U$ is typically used for the universal set. The universal set might change from problem to problem.

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In most areas of mathematics, our reasoning can be aided and clarified by utilizing various kinds of drawings and diagrams. In set theory, we commonly use Venn diagrams, developed by the logician John Venn (1834–1923). In these diagrams, the universal set is represented by a rectangle, and other sets of interest within the universal set are depicted by oval regions, or sometimes by circles or other shapes. In the Venn diagram of Figure 1, the entire region bounded by the rectangle represents the universal set $U$, while the portion bounded by the oval represents set $A$. (The size of the oval representing $A$ is irrelevant.) The colored region inside $U$ and outside the oval is labeled $A'$ (read “$A$ prime”). This set, called the complement of $A$, contains all elements that are contained in $U$ but not contained in $A$.  

![Figure 1](image-url)
### The Complement of a Set

For any set $A$ within the universal set $U$, the **complement** of $A$, written $A'$, is the set of elements of $U$ that are not elements of $A$. That is,

$$A' = \{x \mid x \in U \text{ and } x \notin A\}.$$

**Example 1**  
Let $U = \{a, b, c, d, e, f, g, h\}$, 
$M = \{a, b, e, f\}$, 
$N = \{b, d, e, g, h\}$.

Find each of the following sets.

(a) $M'$ 
Set $M'$ contains all the elements of set $U$ that are not in set $M$. Since set $M$ contains the elements $a$, $b$, $e$, and $f$, these elements will be disqualified from belonging to set $M'$, and consequently set $M'$ will contain $c$, $d$, $g$, and $h$, or $M' = \{c, d, g, h\}$.

(b) $N'$ 
Set $N'$ contains all the elements of $U$ that are not in set $N$, so $N' = \{a, c, f\}$.

Consider the complement of the universal set, $U'$. The set $U'$ is found by selecting all the elements of $U$ that do not belong to $U$. There are no such elements, so there can be no elements in set $U'$. This means that for any universal set $U$, $U' = \emptyset$.

Now consider the complement of the empty set, $\emptyset'$. Since $\emptyset' = \{x \mid x \in U \text{ and } x \notin \emptyset\}$ and set $\emptyset$ contains no elements, every member of the universal set $U$ satisfies this definition. Therefore for any universal set $U$, $\emptyset' = U$.

Suppose that we are given the universal set $U = \{1, 2, 3, 4, 5\}$, while $A = \{1, 2, 3\}$. Every element of set $A$ is also an element of set $U$. Because of this, set $A$ is called a **subset** of set $U$, written

$$A \subseteq U.$$ 

(“$A$ is not a subset of set $U$” would be written $A \nsubseteq U$.) A Venn diagram showing that set $M$ is a subset of set $N$ is shown in Figure 2.

### Subset of a Set

Set $A$ is a **subset** of set $B$ if every element of $A$ is also an element of $B$. 
In symbols,

$$A \subseteq B.$$ 

**Example 2**  
Write $\subseteq$ or $\nsubseteq$ in each blank to make a true statement.  

(a) $\{3, 4, 5, 6\}$ ____ $\{3, 4, 5, 6, 8\}$

Since every element of $\{3, 4, 5, 6\}$ is also an element of $\{3, 4, 5, 6, 8\}$, the first set is a subset of the second, so $\subseteq$ goes in the blank.
(b) \{1, 2, 3\} \text{____} \{2, 4, 6, 8\}

The element 1 belongs to \{1, 2, 3\} but not to \{2, 4, 6, 8\}. Place \(\not\in\) in the blank.

(c) \{5, 6, 7, 8\} \text{____} \{5, 6, 7, 8\}

Every element of \{5, 6, 7, 8\} is also an element of \{5, 6, 7, 8\}. Place \(\subseteq\) in the blank.

As Example 2(c) suggests, every set is a subset of itself:

\[B \subseteq B\quad \text{for any set } B.\]

The statement of set equality in Section 2.1 can be formally presented using subset terminology.

### Set Equality (Alternative definition)
If \(A\) and \(B\) are sets, then \(A = B\) if \(A \subseteq B\) and \(B \subseteq A\).

When studying subsets of a set \(B\), it is common to look at subsets other than set \(B\) itself. Suppose that \(B = \{5, 6, 7, 8\}\) and \(A = \{6, 7\}\). \(A\) is a subset of \(B\), but \(A\) is not all of \(B\); there is at least one element in \(B\) that is not in \(A\). (Actually, in this case there are two such elements, 5 and 8.) In this situation, \(A\) is called a proper subset of \(B\). To indicate that \(A\) is a proper subset of \(B\), write \(A \subset B\).

### Proper Subset of a Set
Set \(A\) is a proper subset of set \(B\) if \(A \subseteq B\) and \(A \neq B\). In symbols,

\[A \subset B.\]

(Notice the similarity of the subset symbols, \(\subset\) and \(\subseteq\), to the inequality symbols from algebra, \(<\) and \(\leq\).)

**Example 3** Decide whether \(\subset\), \(\subseteq\), or both could be placed in each blank to make a true statement.

(a) \{5, 6, 7\} \text{____} \{5, 6, 7, 8\}

Every element of \{5, 6, 7\} is contained in \{5, 6, 7, 8\}, so \(\subseteq\) could be placed in the blank. Also, the element 8 belongs to \{5, 6, 7, 8\} but not to \{5, 6, 7\}, making \{5, 6, 7\} a proper subset of \{5, 6, 7, 8\}. This means that \(\subset\) could also be placed in the blank.

(b) \{a, b, c\} \text{____} \{a, b, c\}

The set \{a, b, c\} is a subset of \{a, b, c\}. Since the two sets are equal, \{a, b, c\} is not a proper subset of \{a, b, c\}. Only \(\subseteq\) may be placed in the blank.

Set \(A\) is a subset of set \(B\) if every element of set \(A\) is also an element of set \(B\). This definition can be reworded by saying that set \(A\) is a subset of set \(B\) if there are
2.2 Venn Diagrams and Subsets

no elements of \( A \) that are not also elements of \( B \). This second form of the definition shows that the empty set is a subset of any set, or
\[
\emptyset \subseteq B \quad \text{for any set } B.
\]
This is true since it is not possible to find any elements of \( \emptyset \) that are not also in \( B \). (There are no elements in \( \emptyset \).) The empty set \( \emptyset \) is a proper subset of every set except itself:
\[
\emptyset \subset B \quad \text{if } B \text{ is any set other than } \emptyset.
\]
Every set (except \( \emptyset \)) has at least two subsets, \( \emptyset \) and the set itself.

**Example 4** Find all possible subsets of each set.

(a) \( \{7, 8\} \)

By trial and error, the set \( \{7, 8\} \) has four subsets:
\[
\emptyset, \{7\}, \{8\}, \{7, 8\}.
\]

(b) \( \{a, b, c\} \)

Here trial and error leads to 8 subsets for \( \{a, b, c\} \):
\[
\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.
\]

In Example 4, the subsets of \( \{7, 8\} \) and the subsets of \( \{a, b, c\} \) were found by trial and error. An alternative method involves drawing a **tree diagram**, a systematic way of listing all the subsets of a given set. Figures 3(a) and (b) show tree diagrams for \( \{7, 8\} \) and \( \{a, b, c\} \).

**FIGURE 3**

In Example 4, we determined the number of subsets of a given set by making a list of all such subsets and then counting them. The tree diagram method also produced a list of all possible subsets. In many applications, we don’t need to display all the subsets but simply determine how many there are. Furthermore, the trial and error method and the tree diagram method would both involve far too much work if the original set had a very large number of elements. For these reasons, it is desirable to have a formula for the number of subsets. To obtain such a formula, we use inductive reasoning. That is, we observe particular cases to try to discover a general pattern.

Begin with the set containing the least number of elements possible—the empty set. This set, \( \emptyset \), has only one subset, \( \emptyset \) itself. Next, a set with one element has only
two subsets, itself and \( \emptyset \). These facts, together with those obtained above for sets with two and three elements, are summarized here.

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of subsets</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

This chart suggests that as the number of elements of the set increases by one, the number of subsets doubles. This suggests that the number of subsets in each case might be a power of 2. Every number in the second row of the chart is indeed a power of 2. Add this information to the chart.

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of subsets</td>
<td>( 1 = 2^0 )</td>
<td>( 2 = 2^1 )</td>
<td>( 4 = 2^2 )</td>
<td>( 8 = 2^3 )</td>
</tr>
</tbody>
</table>

This chart shows that the number of elements in each case is the same as the exponent on the 2. Inductive reasoning gives the following generalization.

**Number of Subsets**

The number of subsets of a set with \( n \) elements is \( 2^n \).

Since the value \( 2^n \) includes the set itself, we must subtract 1 from this value to obtain the number of proper subsets of a set containing \( n \) elements.

**Number of Proper Subsets**

The number of proper subsets of a set with \( n \) elements is \( 2^n - 1 \).

As shown in the chapter on problem solving, although inductive reasoning is a good way of discovering principles or arriving at a conjecture, it does not provide a proof that the conjecture is true in general. A proof must be provided by other means. The two formulas above are true, by observation, for \( n = 0, 1, 2, \) or 3. (For a general proof, see Exercise 71 at the end of this section.)

**Example 5** Find the number of subsets and the number of proper subsets of each set.

(a) \( \{3, 4, 5, 6, 7\} \)

This set has 5 elements and \( 2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32 \) subsets. Of these, \( 2^5 - 1 = 32 - 1 = 31 \) are proper subsets.

(b) \( \{1, 2, 3, 4, 5, 9, 12, 14\} \)

This set has 8 elements. There are \( 2^8 = 256 \) subsets and 255 proper subsets.