IA-automorphisms and localization of nilpotent groups

by

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Abstract

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Advisor: Professor Joseph Roitberg

A group is called \( p \)-local, where \( p \) is a prime number, if every element in the group has a unique \( n \)th root for each \( n \) relatively prime to \( p \). Given a nilpotent group \( G \) and a prime \( p \), there is a unique \( p \)-local group \( G(p) \) which is, in some sense, the “best approximation” to \( G \) among all \( p \)-local nilpotent groups. \( G(p) \) is called the \( p \)-localization of \( G \).

Let \( G(p) \) be the \( p \)-localization of a nilpotent group \( G \), and let \( IA(G) \) be the subgroup of \( AutG \) consisting of those automorphisms of \( G \) that induce the identity on \( G/G' \), where \( G' \) denotes the commutator subgroup of \( G \). \( IA(G) \) turns out to be nilpotent, so its \( p \)-localization exists. Two groups \( G \) and \( H \) are said to be in the same localization genus if \( G(p) \) is isomorphic to \( H(p) \) for all primes \( p \). The main result of this thesis is that if two finitely generated, torsion-free, nilpotent, and metabelian groups lie in the same localization genus, their \( IA \)-groups also lie in the same localization genus. The method of proof involves basic sequences and commutator calculus.
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Chapter 1

Background

1.1 Nilpotent groups and basic sequences

We begin with a brief account on nilpotent groups and basic commutators. For a detailed presentation of this material, refer to [2] and [16].

Let $G$ be any group. The lower central series of $G$ is defined as:

$$G = \gamma_1 G \geq \gamma_2 G \geq \cdots$$

where $\gamma_i G$ is the normal subgroup of $G$ generated by all elements of $G$ of the form $[g_1, \ldots, g_i]$. Here,

$$[g_1, \ldots, g_i] = [[g_1, \ldots, g_{i-1}], g_i], \quad [g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2.$$  

$G$ is called nilpotent if $\gamma_{c+1} G = 1$ for some $c$. In case $\gamma_c G \neq 1$, we say that $G$ is nilpotent of class $c$.

Lemma 1.1.1. The following commutator identities hold in any group:

1. $[x, yz] = [x, z] [x, y]^z$,  

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2. \([xy, z] = [x, z]^y [y, z],\)

3. \([x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = 1,\)

where \(a^b = b^{-1}ab.\)

Proof. The first two identities are straightforward so we only verify the third. Let

\[ u = xzx^{-1}yx, \quad v = yxy^{-1}zy, \quad \text{and} \quad w = zy(z^{-1}xz). \]

(v and w are obtained from u as cyclic permutations of x, y, and z). From

\[ [x, y^{-1}, z] = [y^{-1}, x] [x, y^{-1}] [x, y^{-1}, z] = [y^{-1}, x] [x, y^{-1}]^z, \]

it is easy to check that

\[ [x, y^{-1}, z]^y = u^{-1}v. \]

Similarly,

\[ [y, z^{-1}, x]^z = v^{-1}w, \]

and

\[ [z, x^{-1}, y]^x = w^{-1}u. \]

The lemma now follows from this. \(\square\)

In order to construct a sequence of basic commutators in a nilpotent group we consider a more general setup, an approach taken in [2]. Let

\[ X = \{x_1, \ldots, x_q\} \]

be a finite set. We define the free groupoid \(G, \) freely generated by \(X,\) as the set of all “bracketed” finite words on \(X,\) under a binary operation called
bracket. Observe that if \( g \) and \( h \) lie in \( G \), their bracket \([g, h]\) also lies in \( G \).

Observe further that \( G \) does not satisfy associativity, commutativity, or any other group-theoretic law, except closure under the bracket.

For example, take \( q = 3 \). Then \( x_1, x_2 \in G \), \((x_1 x_2) x_1 \in G\), \(((x_1 x_2) x_1) (x_2 x_3) \in G\), and so on.

The length of \( g \in G \), denoted \(|g|\), is the number of letters appearing in \( g \).

**Definition 1.1.2.** Let \( G \) be a free groupoid on \( X \). Let \( b_1, b_2, \ldots \) be an infinite sequence of elements in \( G \). This sequence is called basic if the following holds:

1. The elements of \( X \) appear in the sequence.
2. If \(|b_i| < |b_j|\) then \( i < j \).
3. Let \( u = vw \in G \) be of length at least 2. Then \( u \) belongs to the basic sequence if and only if
   
   \( a \) \( v = b_i, w = b_j, \text{ and } j < i; \text{ and} \)
   
   \( b \) \( \text{either } |v| = 1 \text{ (in which case } |w| = 1 \text{ by the previous items) or} \)
   
   \( v = b_k b_l \text{ where } l \leq j \).

To construct a basic sequence, we define the so-called “rep” operation. Let \( A \subset G \) and \( a \in A \). Define

\[
A \text{ rep } a := \{(bia) : i = 0, 1, \ldots; b \in A - \{a\}\} \subset G
\]

where, by definition,

- \( b0a = b, \) and

- \( b(i + 1)a = (bia) a \) for \( i \geq 0 \).
Consider the free groupoid $G$ on $X = \{x_1, \ldots, x_q\}$. We construct a sequence of subsets of $G$ as follows:

- Put $X_1 = X$. Choose $b_1 \in X_1$.
- Put $X_2 = X_1 \text{ rep } b_1$.

Suppose we have constructed $X_n$. Choose $b_n \in X_n$ of minimal length and put

$$X_{n+1} = X_n \text{ rep } b_n.$$ 

Then $b_1, b_2, \ldots$ is a basic sequence on $X$.

In general, the first $q$ terms of the sequence can be chosen to be $x_1, \ldots, x_q$.

Let $\Gamma$ be a finitely generated groupoid (not necessarily free) with generating set $\{\mu_1, \ldots, \mu_q\}$. We now explain what is meant by a basic sequence on $\Gamma$.

**Definition 1.1.3.** A sequence $\beta_1, \beta_2, \ldots$ of elements of $\Gamma$ is called a basic sequence on $\{\mu_1, \ldots, \mu_q\}$ if and only if there exists a basic sequence $b_1, b_2, \ldots$ on $X$ such that the groupoid homomorphism

$$G \to \Gamma$$

$$x_i \mapsto \mu_i$$

sends $b_i$ to $\beta_i$.

Let $G$ be a group, generated by the finite set

$$Y = \{y_1, \ldots, y_q\}.$$
Ignoring the fact that $G$ is a group, we can construct a groupoid relative to the binary operation “commutation”:

$$[g,h] = g^{-1}h^{-1}gh.$$ 

Note that $Y$ no longer generates $G$ as a groupoid, but rather, it generates (under commutation) a subgroupoid of $G$.

Use the “rep” operation (as before) to construct a basic sequence $c_1, c_2, \ldots$ on $Y$. We call the terms of this sequence basic commutators on $Y$. We define the weight of $c_i$ as $|b_i|$ where $b_i$ is the “canonical” pre-image of $c_i$ in the free groupoid on $X$.

The first of the following results can be found in [7], and the other two in [15]:

**Theorem 1.1.4.** Let $G$ be a group generated by $Y = \{y_1, \ldots, y_q\}$ and let 

$$c_1, c_2, \ldots$$

be a basic sequence on $Y$.

Then $\gamma_r G/\gamma_{r+1} G$ is generated by the basic commutators of weight $r$ ($r = 1, 2, \ldots$).

**Corollary 1.1.5.** Let $G$ be a group generated by $Y = \{y_1, \ldots, y_q\}$. $G$ is nilpotent if and only if all but finitely many terms on any basic sequence on $Y$ are equal to $1$.

**Theorem 1.1.6.** Let $F$ be a free nilpotent group, freely generated by $X = \{x_1, \ldots, x_q\}$. Let 

$$b_1, b_2, \ldots$$
be a basic sequence on $X$. Then $\gamma_r F/\gamma_{r+1} F$ is free abelian on the basic commutators of weight $r$.

## 1.2 IA groups of nilpotent groups

For any group $G$, define $IA(G)$ to be the subgroup of $AutG$ consisting of all automorphisms of $G$ inducing the identity on $G/G'$ where $G' = \gamma_2 G$. We now discuss a series of fundamental results from [7], leading to a major fact about $IA(G)$, when $G$ is nilpotent.

**Lemma 1.2.1.** Let $X$, $Y$, and $Z$ be subgroups of $G$. Let $X^* = [Y, Z, X]$, $Y^* = [Z, X, Y]$, and $Z^* = [X, Y, Z]$. If $N$ is normal in $G$ and both $X^*$ and $Y^*$ are subgroups of $N$, then $Z^*$ is a subgroup of $N$.

**Proof.** A typical generator of $Z^*$ can be written as $[x, y^{-1}, z]$. From lemma 1.1.1, we have:

$$[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = 1.$$ 

Since $N$ is normal in $G$, and both $X^*$ and $Y^*$ are subgroups of $N$, this means that $[x, y^{-1}, z] \in N$, so that $Z^*$ is a subgroup of $N$. 

**Theorem 1.2.2.** Let $H$ and $K$ be subgroups of a group $G$. Let

$$H = H_0 \geq H_1 \geq \ldots$$

be a normal series for $H$ such that

$$[H_i, K] \leq H_{i+1}$$

for each $i$. 


Put $K = K_1$. Define

$$K_j = \{x \in K : [H_i, x] \leq H_{i+j}\}$$

for all $i$. Then

1. $[K_j, K_l] \leq K_{j+l}$ for all $j, l$;
2. $[H_i, \gamma_j K] \leq H_{i+j}$ for all $i, j$.

Proof. We show first that $[K_j, K_l] \leq K_{j+l}$. The second part will follow from this. By definition, $[H_i, K_j] \leq H_{i+j}$ and $[H_{i+j}, K_l] \leq H_{i+j+l}$. Hence

$$[H_i, K_j, K_l] \leq [H_{i+j}, K_l] \leq H_{i+j+l}. \quad (1.1)$$

Likewise,

$$[K_l, H_i, K_j] = [H_i, K_l, K_j] \leq H_{i+j+l}. \quad (1.2)$$

By hypothesis, $H_{i+j+l}$ is normal in $H$. Moreover, for $x \in K$

$$x^{-1}H_{i+j+l}x = H_{i+j+l} [H_{i+j+l}, x].$$

Since $[H_{i+j+l}, x] \subset H_{i+j+l+1}$, we conclude that

$$x^{-1}H_{i+j+l}x \leq H_{i+j+l}.$$  

This shows that $H_{i+j+l}$ is normal in the subgroup of $G$ generated by $H$ and $K$.

Applying lemma 1.2.1 to this situation we get:

$$[K_j, K_l, H_i] = [H_i, [K_j, K_l]] \leq H_{i+j+l}. \quad (1.3)$$
Therefore, by definition,
\[ [K_j, K_l] \leq K_{j+l}. \]  
(1.4)

In particular, \([K_j, K] \leq K_{j+1}\) so

\[ K = K_1 \geq K_2 \geq \cdots \]
is a central series for \(K\). This implies that \(\gamma_j K \leq K_j\) and hence

\[ [H_i, \gamma_j K] \leq [H_i, K_j] \leq H_{i+j}. \]  
(1.5)

This completes the proof.

\[ \square \]

**Corollary 1.2.3.** For any group \(G\),

\[ [\gamma_i, \gamma_j] \leq \gamma_{i+j}. \]

**Proof.** This follows from theorem 1.2.2 by taking \(H_i = \gamma_{i+1}\) and \(K = G\). \[ \square \]

In order to express what follows in the correct language we need to construct the so-called “holomorph” of a group \(G\) (see \([10]\)). Let

\[ G^* = \{ \phi g : \phi \in \text{Aut}G, \ g \in G \}. \]

\(G^*\) can be regarded as the cartesian product

\[ \text{Aut}G \times G. \]

This set becomes a group under the operation

\[ (\phi g)(\phi' g') = \phi\phi' g\phi' g', \]
where $g^{\phi'} = \phi'(g) \in G$. ($G^*$ is in fact a semi-direct product of $G$ by $\text{Aut}G$).

The notation

$$G^* = \text{Hol}G$$

is customary, and we call $\text{Hol}G$ the holomorph of $G$.

**Lemma 1.2.4.** Let

$$G = G_0 \geq G_1 \ldots \geq G_r = 1$$

be a series of normal subgroups in $G$.

Let $A$ be the group of automorphisms of $G$ leaving each $G_i$ invariant and transforming

$$G_i/G_{i+1}$$

identically. Then $A$ and $[G, A]$ are nilpotent of class less than $r$. Here, $[G, A] \leq \text{Hol}G$.

**Proof.** Notice that $[G_i, A] \leq G_{i+1}$ for each $i$: if $x \in G_i$ and $\alpha \in A$, we have

$$[x, \alpha] = x^{-1}\alpha^{-1}x\alpha = x^{-1}x^\alpha \in G_{i+1}$$

since $\alpha$ transforms $G_i/G_{i+1}$ identically. By theorem 1.2.2 with $H = G$ and $A = K$,

$$[G, \gamma_r A] = [G_0, \gamma_r A] \leq G_r = 1;$$

so that

$$[G, \gamma_r A] = 1.$$ 

Now let $\alpha \in \gamma_r A$ and $x \in G$. Then

$$[x, \alpha] = x^{-1}x^\alpha = 1,$$
which shows that $\alpha = 1$. As a consequence, $\gamma_r A = 1$ so $A$ is nilpotent of class less than $r$.

To show that $[G, A]$ is nilpotent of class less that $r$ we proceed as follows: since $G_{i-1}$ is normal in $G$, then $[G_{i-1}, G] \leq G_{i-1}$. Hence

$$[G_{i-1}, G, A] \leq [G_{i-1}, A] \leq G_i.$$  

Also,

$$[A, G_{i-1}, G] \leq [G_i, G] \leq G_i.$$  

By assumption, $x^\alpha \in G_i$ for $x \in G_i$ and $\alpha \in A$, so that $G_i$ is normal in the subgroup of $HolG$ generated by $A$ and $G$. We can then apply lemma 1.2.1 to obtain

$$[G, A, G_{i-1}] = [G_{i-1}, [G, A]] \leq G_i.$$  

Thus, by theorem 1.2.2,

$$[G_1, \gamma_{r-1}[G, A]] \leq G_r = 1.$$  

But

$$[G, A] = [G_0, A] \leq G_1,$$  

so that

$$[[G, A], \gamma_{r-1}[G, A]] = 1.$$  

By definition of the lower central series, $\gamma_r [G, A] = 1$, as required.

\[\square\]

**Lemma 1.2.5.** Let $H$ and $K$ be subgroups of a group $G$ such that

$$[H, K] \leq H'.$$  

Then

$$[\gamma_i H, \gamma_j K] \leq \gamma_{i+j} H$$  

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for all \(i, j\).

**Proof.** We do “double induction.” Assume \(j = 1\). We prove that

\[
[\gamma_i H, K] \leq \gamma_{i+1} H
\]

by induction on \(i\). The case \(i = 1\) follows from the hypothesis. Assume that

\[
[\gamma_{i-1} H, K] \leq \gamma_i H.
\]

Notice that:

\[
[\gamma_{i-1} H, K, H] \leq [\gamma_i H, H] = \gamma_{i+1} H \quad \text{(by the induction hypothesis), and}
\]

\[
[K, H, \gamma_{i-1} H] \leq [\gamma_2 H, \gamma_{i-1} H] \leq \gamma_{i+1} H \quad \text{(by hypothesis and corollary 1.2.3)}.
\]

Using lemma 1.2.1 and the definition of \(\gamma_i H\),

\[
[\gamma_i H, K] = [H, \gamma_{i-1} H, K] \leq \gamma_{i+1} H.
\]

The basis of induction for \(j\) now follows.

The induction hypothesis for \(j\) is

\[
[\gamma_i H, \gamma_{j-1} K] \leq \gamma_{i+j-1} H
\]

for all \(i\). Note that

\[
[\gamma_i H, \gamma_j K] = [\gamma_i H, [K, \gamma_{j-1} K]] = [K, \gamma_{j-1} K, \gamma_i H].
\]

By the induction hypothesis on \(j\):

\[
[\gamma_i H, K, \gamma_{j-1} K] \leq [\gamma_{i+1} H, \gamma_{j-1} K] \leq \gamma_{i+j} H \quad \text{and}
\]

\[
[\gamma_{j-1} K, \gamma_i H, K] \leq [\gamma_{i+j-1} H, K] \leq \gamma_{i+j} H.
\]
Using lemma 1.2.1 once again, we conclude that

\[ [\gamma_i H, \gamma_j K] \leq \gamma_{i+j} H. \]

This completes the double induction. \( \square \)

**Corollary 1.2.6.** Let \( H \) be nilpotent of class \( c \). Then

- \( \gamma_j IA(H) \) transforms each \( \gamma_i H/\gamma_{i+j} H \) identically, and
- \( IA(H) \) is nilpotent of class \( c - 1 \).

**Proof.** To prove the first assertion, notice that if we choose \( x \in H \) and \( \psi \in IA(H) \), then

\[ [x, \psi] = x^{-1} x^\psi \in H', \]

in \( HolH \). Hence

\[ [H, IA(H)] \leq H' \]

and by lemma 1.2.5,

\[ [\gamma_i H, \gamma_j IA(H)] \leq \gamma_{i+j} H. \]

This means that \( \gamma_j IA(H) \) transforms each \( \gamma_i H/\gamma_{i+j} H \) identically.

To see that \( IA(H) \) is nilpotent of class less than \( c \), let \( \alpha \in \gamma_c IA(H) \) and \( x \in H = \gamma_i H \). By the first assertion, \( \alpha \) transforms \( \gamma_1 H/\gamma_{c+1} H \cong H \) identically. This proves that \( \gamma_c IA(H) = 1 \) so that \( IA(H) \) is nilpotent of class less than \( c \).

Finally recall that the inner automorphisms of \( H \) constitute a subgroup of \( IA(H) \) which is isomorphic to \( H \) modulo its center. This shows that \( IA(H) \) must have class exactly \( c - 1 \). \( \square \)
We end this section with additional results about $IA(G)$. We state and prove some pertinent facts about nilpotent groups first.

**Lemma 1.2.7.** The upper central quotients of a torsion-free nilpotent group are torsion-free.

**Proof.** Let 
\[ \zeta_i = \{g \in G : [g, x] \in \zeta_{i-1} \text{ for all } x \in G\} \]
be the $i$-th center of $G$; where $\zeta_1 = \zeta$, the center of $G$. The proof is by induction on $i$, the basis of induction being obvious since $\zeta$ is also torsion-free. Assume $\zeta_{i-1}/\zeta_{i-2}$ is torsion-free. We will show that so is $\zeta_i/\zeta_{i-1}$. Let $g \in \zeta_i$ and $m > 0$ such that $g^m \in \zeta_{i-1}$. We wish to show that $g \in \zeta_{i-1}$. Let $x \in G$. Then
\[
1 = [g^m, x] = [gg^{m-1}, x] = [g, x] [g, x, g^{m-1}] [g^{m-1}, x] = [g, x] [g^{m-1}, x] \pmod {\zeta_{i-2}}.
\]
Continue reducing in this way to finally obtain
\[
1 = [g^m, x] = [g, x]^m \pmod {\zeta_{i-2}}.
\]
By the induction hypothesis, $[g, x] \in \zeta_{i-2}$ for every $x$, so that $g \in \zeta_{i-1}$, as required.

**Corollary 1.2.8.** $G/\zeta$ is torsion-free.

**Proof.** Let $g \in G$ and $g^m \in \zeta$ with $m > 0$. Since $G$ is nilpotent there is an $i$ such that $g \in \zeta_i$. The fact that $\zeta \leq \zeta_{i-1}$ yields $g^m \in \zeta_{i-1}$. By lemma 1.2.7, $g$ itself belongs to $\zeta_{i-1}$. Continue this argument down the upper central series of $G$ to finally conclude that $g \in \zeta$. This gives that $G/\zeta$ is torsion-free.
The fact that $G/\zeta$ is isomorphic to the group of inner automorphisms of $G$, suggests that $IA(G)$ may be torsion-free. This is in fact true. We have:

**Lemma 1.2.9.** If $G$ is a torsion-free nilpotent group, $IA(G)$ is torsion-free.

*Proof.* Define

$$\gamma_c = \{g \in G : \text{there exists } n > 0 \text{ with } g^n \in \gamma_c G\},$$

where $G$ is a torsion-free nilpotent group of class $c$. It is immediate that $\gamma_c$ is a normal and central subgroup of $G$ containing $\gamma_c$. $G/\gamma_c$ is nilpotent of class less than $c$ since $c$-fold commutators are trivial in $G/\gamma_c$. Moreover, $G/\gamma_c$ is torsion-free, for if $g \in G$ with $g^m \in \gamma_c$, there is an $n$ such that $g^{nm} \in \gamma_c$; hence $g \in \gamma_c$.

The proof that $IA(G)$ is torsion-free is by induction on the class of $G$. If $G$ has class 2, $IA(G)$ is abelian. Let $\varphi \in IA(G)$ and $x$ be an arbitrary generator of $G$. Then

$$\varphi(x) = xd$$

where $d \in G'$. Suppose $\varphi^m = 1$ where $m > 0$. Then

$$\varphi^m(x) = xd^m = x,$$

and by torsion-freeness of $G'$, $d = 1$. This completes the basis of induction.

Assume now that the $IA$-group of a torsion-free nilpotent group of class less that $c$ is always torsion-free. Let $G$ be of class $c$. Since $G/\gamma_c$ is nilpotent, torsion-free, and of class less than $c$, the induction hypothesis gives that $IA(G/\gamma_c)$ is torsion-free. To prove that $IA(G)$ is torsion-free, let $\varphi \in IA(G)$
and assume that $\varphi^m = 1$ where $m > 0$. Consider the canonical homomorphism 

$$IA(G) \to IA(G/\gamma_c),$$

$$\tau \mapsto \hat{\tau}$$

where 

$$\hat{\tau}((g)) = \langle \tau(g) \rangle.$$ 

For $x \in G$, $\langle x \rangle$ denotes the equivalence class of $x$ in $G/\gamma_c$. Since $\varphi^m$ is the identity and $IA(G/\gamma_c)$ is torsion-free, $\hat{\varphi}$ is the identity. Let $x$ be an arbitrary generator of $G$ and write 

$$\varphi(x) = xd$$

where $d \in G'$. Then 

$$\hat{\varphi}(\langle x \rangle) = \langle \varphi(x) \rangle = \langle xd \rangle = \langle x \rangle.$$ 

Hence 

$$d \in \gamma_c.$$ 

We claim that $\varphi$ acts trivially on $\gamma_c$. Let $y$ be any element in $\gamma_c$. There exists $y_1 \in G'$ such that 

$$\varphi(y) = yy_1.$$ 

Moreover, there is a positive integer $m$ such that $y^m \in \gamma_c$. Hence 

$$y^m = \varphi(y^m) = \varphi(y)^m = (yy_1)^m = y^m y_1^m$$

(recall that $\gamma_c$ is central). Since $G$ is torsion-free, this means $y_1 = 1$, so that $\varphi$ acts trivially on $\gamma_c$. This, together with $\varphi(x) = xd$ and $\varphi^m = 1$ gives 

$$\varphi^m(x) = x d^m = x,$$
so that $d^n = 1$, and since $G$ is torsion-free, $d = 1$. This completes the proof.

Lemma 1.2.10. If $G$ is a finitely generated nilpotent group, $IA(G)$ is finitely generated.

Proof. The proof is by induction on the class of $G$. If $G$ has class 2, each element of $IA(G)$ acts trivially on $G'$ and $IA(G)$ is abelian. A typical member of a generating set for $IA(G)$ can be constructed as follows: for each generator $x_i$ of $G$ choose a generator $y_j$ of $G'$. Construct the $IA$-automorphism that sends $x_i$ to $x_iy_j$ and each remaining generator of $G$ to itself. Since the generating sets for $G$ and $G'$ can be chosen to be finite, this generating set for the abelian group $IA(G)$ will also be finite.

Assume the induction hypothesis: the $IA$-group of finitely generated nilpotent groups of class less than $c$ is finitely generated.

Let $G$ be of class $c$. Consider the following subgroup of $IA(G)$:

$$I_c = \{ \alpha \in IA(G) : g^{-1} \alpha(g) \in \gamma_c \text{ for all } g \in G \}.$$

If $\alpha \in I_c$, then

$$\alpha(x_i) = x_i h_i$$

where $x_i$ is a typical generator of $G$ and $h_i \in \gamma_c$. It follows from this that the elements of $I_c$ act trivially on $G'$ and $I_c$ is in fact an abelian subgroup of $IA(G)$.

Consider the natural homomorphism

$$\phi : IA(G) \to IA(G/\gamma_c)$$

$$\varphi \mapsto \hat{\varphi}$$
where
\[ \hat{\phi}(\langle g \rangle) = \langle \varphi(g) \rangle. \]

We prove that

1. \( I_c = \ker \phi \), and

2. \( I_c \) is finitely generated.

The fact that \( I_c = \ker \phi \) follows from the definition of \( I_c \). The fact that \( I_c \) is finitely generated can be established by an analogous construction as in the basis of induction: for each generator \( x_i \) of \( G \) choose a generator \( y_j \) of \( \gamma_c \). Construct the \( IA \)-automorphism that sends \( x_i \) to \( x_iy_j \) and each remaining generator of \( G \) to itself. Since these generating sets are finite, the generating set for \( I_c \) so obtained is also finite.

By our induction hypothesis, \( IA(G/\gamma_c) \) is finitely generated so the image of \( \phi \) (being a subgroup of a finitely generated nilpotent group) is also finitely generated. This image is isomorphic to \( IA(G)/I_c \).

Finally, a set consisting of one representative for each equivalence class of \( IA(G)/I_c \), together with the generating set for \( I_c \) gives a generating set for \( IA(G) \). This completes the proof.

\[ \square \]

**Remark.** Assuming nilpotency of \( G \) is essential here. For example, if \( G \) is a two generator metabelian group, C.K. Gupta proved that \( IA(G) \) is still metabelian (see reference [7] in [1]). However, \( IA(G) \) need not be finitely generated (see theorem C in [1]).

**Corollary 1.2.11.** If \( G \) is finitely generated, torsion-free nilpotent of class \( c \), \( IA(G) \) is finitely generated, torsion-free nilpotent of class \( c - 1 \).
Proof. This follows from lemmas 1.2.9 and 1.2.10 together with corollary 1.2.6.

It is well known that if $G$ is finitely generated, torsion-free nilpotent, then any Eilenberg-MacLane $K(G, 1)$ space has the homotopy type of a finite complex. Our results now imply:

**Corollary 1.2.12.** Let $G$ be a finitely generated, torsion-free, nilpotent group, then any $K(IA(G), 1)$ space has the homotopy type of a finite complex.

### 1.3 Localization and completion of nilpotent groups

Refer to [9] and [16] for fine accounts of the material on localization. We use the following notation:

- $P$ denotes a set of primes.
- $P'$ denotes the set of primes not in $P$.
- $n \in P'$ means that the natural number $n$ only involves primes from $P'$.

**Definition 1.3.1.** A group $G$ is called $P$-local if and only if the map

$$G \to G$$

$$x \mapsto x^n$$

is a bijection for all $n \in P'$. 
Definition 1.3.2. Let $\mathcal{H}$ be a subcategory of the category of groups. A morphism

$$e : G \to G_P$$

in $\mathcal{H}$ is said to be $P$-universal or a $P$-localizing map if

1. $G_P$ is $P$-local.
2. For any $P$-local group $K \in \mathcal{H}$, the map:

$$e^* : \text{Hom}(G_P, K) \to \text{Hom}(G, K)$$

$$\varphi \mapsto e^*(\varphi)$$

where

$$e^*(\varphi)(g) = \varphi e(g).$$

is a bijection.

Assume next that each group in $\mathcal{H}$ admits a $P$-localizing map. Given a morphism

$$\varphi : G \to K$$

in $\mathcal{H}$, there exists a unique morphism

$$\varphi_P : G_P \to K_P$$

making the diagram

$$\begin{array}{ccc}
G & \overset{\varphi}{\longrightarrow} & K \\
\downarrow^e & & \downarrow^e \\
G_P & \overset{\varphi_P}{\longrightarrow} & K_P
\end{array}$$

commute. Here, $e$ denotes the localization map.

The above diagram gives a functor $L$ from $\mathcal{H}$ to itself. The pair $(L, e)$ is called a localization theory in $\mathcal{H}$. 

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Definition 1.3.3. Let $G$ be any group. An element $x \in G$ is said to be $P'$-torsion if there is an $n \in P'$ such that $x^n = 1$.

Definition 1.3.4. A homomorphism

$$\varphi : G \rightarrow K$$

is called $P$-injective if

$$\ker \varphi = \{x \in G : x \text{ is } P'-\text{torsion}\}.$$

Definition 1.3.5. A homomorphism

$$\varphi : G \rightarrow K$$

is called $P$-surjective if for all $y \in K$ there exists $n \in P'$ such that $y^n$ lies in the image of $\varphi$.

Definition 1.3.6. A $P$-isomorphism is a homomorphism which is $P$-injective and $P$-surjective.

Refer to [9] for proofs of the following results.

Lemma 1.3.7. Let $\varphi : G \rightarrow G'$ be a homomorphism of $P$-local groups.

1. If $\varphi$ is $P$-injective then $\varphi$ is one-to-one.

2. If $\varphi$ is $P$-surjective then $\varphi$ is onto.

Fundamental theorem of $P$-localization of nilpotent groups

For a nilpotent group $G$, write $\text{nil}G$ for its nilpotency class.
1. There exists a localization theory \((L,e)\) in the category of nilpotent groups \(\mathcal{N}\).

2. If \(c \geq 1\), \((L,e)\) restricts to a localization theory in \(\mathcal{N}_c\), the category of nilpotent groups of class at most \(c\).

3. From (2),

\[ \text{nil}LG \leq \text{nil}G \]

where \(G \in \mathcal{N}\).

4. Let \(\varphi : G \to K\) be a morphism in \(\mathcal{N}\). \(\varphi\) is a \(P\)-localizing map if and only if

(a) \(K\) is \(P\)-local, and
(b) \(\varphi\) is a \(P\)-isomorphism.

The last item is a very useful fact.

**Theorem 1.3.8.** Let

\[
1 \to G' \to G \to G'' \to 1
\]

be a short exact sequence of nilpotent groups. If any two of these groups are \(P\)-local, so is the third.

**Theorem 1.3.9.** Let

\[
\begin{array}{cccccc}
1 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & G'' & \longrightarrow & 1 \\
\downarrow \varphi' & & \downarrow \varphi & & \downarrow \varphi'' & & \\
1 & \longrightarrow & H' & \longrightarrow & H & \longrightarrow & H'' & \longrightarrow & 1
\end{array}
\]

be a map of short exact sequences of nilpotent groups. If any two of \(\varphi', \varphi, \varphi''\) \(P\)-localizes, so does the third.
Theorem 1.3.10. P-localization is an exact functor. That is, applying P-localization to a short exact sequence of nilpotent groups

\[ 1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1 \]

yields a short exact sequence of P-local nilpotent groups

\[ 1 \rightarrow G'_p \rightarrow G_p \rightarrow G''_p \rightarrow 1. \]

If P consists of a single prime p, we write \( G_p \) for the P-localization of G (soon we will discuss \( G_p \), the “p-completion” of G). For a single prime p, a homomorphism may be p-injective, p-surjective, or a p-isomorphism.

Definition 1.3.11. Two nilpotent groups \( G \) and \( H \) are said to be in the same localization genus if they are p-isomorphic for every prime p.

A concept related to p-localization is that of p-completion. To define the p-completion of a nilpotent group we recall the notion of inverse systems of groups and their limits (see [6]). A quasi-order in a set \( M \) is a relation on \( M \) which is reflexive and transitive. In general, the relation need not be symmetric. A set \( M \) furnished with a quasi-order “<” is called directed if for every \( \alpha, \beta \) in \( M \), there exists \( \gamma \) in \( M \) such that \( \alpha < \gamma \) and \( \beta < \gamma \).

Let \( M \) be a directed set. An inverse system of groups over \( M \) is an assignment \( \{ G, \Pi \} \) such that for each \( \alpha \in M \), \( G_\alpha \) is a group and for each \( \alpha < \beta \),

\[ \pi^\beta_\alpha : G_\beta \rightarrow G_\alpha \]

is a group homomorphism satisfying:

1. \( \pi^\alpha_\alpha \) is the identity for all \( \alpha \in M \) and
The homomorphisms $\pi^\beta_\alpha$ are called projections of the system.

The inverse limit of the system \( \{ G, \Pi \} \), denoted \( G_\infty \), is the subgroup of the product \( \Pi_{\alpha \in M} G_\alpha \) consisting of those functions \( g = \{ g_\alpha \} \) satisfying 

$\pi^\beta_\alpha (g_\beta) = g_\alpha$ for each \( \alpha < \beta \).

Let \( G \) be a nilpotent group, \( p \) a prime, and \( N \) the set of non-negative integers. Then \( N \) is a directed set under the relation “\( \leq \).” For each \( \alpha \in N \), the subgroup

$$G^{p^\alpha} = gp\left( g^{p^\alpha} : g \in G \right)$$

is normal in \( G \). Let

$$G_\alpha = G / G^{p^\alpha}$$

and

$$\pi^\beta_\alpha : G_\beta \to G_\alpha,$$

for \( \alpha < \beta \) be the obvious homomorphism. Each projection clearly satisfies the above conditions, so \( \{ G, \Pi \} \) is an inverse system of \( p \)-groups. We call the nilpotent group \( G_\infty \) the \( p \)-adic completion of \( G \) (or \( p \)-completion for short) and use the notation \( G_\infty = G_p \).

**Remark.** If \( G \) is finitely generated, the \( G_\alpha \) are torsion, finitely generated nilpotent groups, so each \( G_\alpha \) is a finite \( p \)-group.

For every nilpotent group \( G \), there is a natural homomorphism \( G \to G_p \) with kernel

$$G^{p^\omega} = \bigcap_{\alpha \in N} G^{p^\alpha}.$$

If \( G \) is finitely generated and torsion-free, \( G_p \) is also torsion-free and the natural homomorphism \( G \to G_p \) is injective [12].
**Definition 1.3.12.** Two groups are in the same completion genus if their \(p\)-completions are isomorphic for every \(p\).

The following results will help us compare the localization genus with the completion genus.

**Lemma 1.3.13.** Let \(G\) be a torsion nilpotent group. Then \(G\) is \(p\)-local if and only if \(G\) has no \(n\)-torsion for \(n\) relatively prime to \(p\).

**Proof.** Suppose first that \(G\) is \(p\)-local. For \(n\) relatively prime to \(p\), \(x \mapsto x^n\) is a bijection. Hence \(G\) has no \(n\)-torsion. Conversely, suppose \(G\) has no \(n\)-torsion for all \(n\) relatively prime to \(p\) and consider the localization map

\[e : G \to G_p.\]

We show first that \(e\) is 1-1: since \(e\) is \(p\)-injective, if \(g \in \ker e\) there exists \(n\), relatively prime to \(p\), such that \(g^n = 1\). As \(G\) has no \(n\)-torsion it follows that \(g = 1\). Hence \(e\) is injective.

Next, we show that "\(G_p\) is a torsion group." Let \(g \in G_p\). There exist \(n\) relatively prime to \(p\) and \(x \in G\) such that

\[e(x) = g^n.\]

Since \(G\) is a torsion group, but has no \(n\)-torsion for \(n\) relatively prime to \(p\), there are integers \(m\) and \(\alpha\) such that \(x^{mp^\alpha} = 1\) with \((m, p) = 1\). Hence

\[1 = e(x^{mp^\alpha}) = e(x)^{mp^\alpha} = g^{mp^\alpha}\]

in \(G_p\). As \(G_p\) is \(p\)-local, \(g^{mp^\alpha} = 1\). This shows that \(G_p\) is a torsion group.

Let \(t(H)\) be the torsion subgroup of a nilpotent group \(H\). We thus have

\[t(G_p) = G_p.\]
Since all of the torsion elements of a localization come from the original group (corollary 8.6 in [16]),

\[ G_{(p)} = t(G_{(p)}) = e(t(G)) = e(G), \]

so that \( e \) is onto. This proves that \( e \) is an isomorphism and \( G \) is \( p \)-local. \( \square \)

**Corollary 1.3.14.** Every \( p \)-group is \( p \)-local.

**Theorem 1.3.15.** For any nilpotent group \( G \)

\[ (G_{(p)})_p \cong G_p. \]

**Proof.** \( G_p \) is the inverse limit of (the inverse system of) its \( p \)-group quotients

\[ G/G^{p^\alpha} \]

for \( \alpha \in N \). Similarly, \( (G_{(p)})_p \) (the \( p \)-completion of the \( p \)-localization of \( G \)) is the inverse limit of (the inverse system of) its \( p \)-group quotients

\[ G_{(p)}/G^{p^\alpha}_{(p)} \]

for \( \alpha \in N \). Since \( p \)-groups are \( p \)-local, we have that for each \( \alpha \),

\[ G/G^{p^\alpha} \cong (G/G^{p^\alpha})_{(p)}. \]

To show that the inverse systems of \( G \) and \( G_{(p)} \) agree, and their corresponding limits are therefore isomorphic, it suffices to check that the \( p \)-localization of \( G/G^{p^\alpha} \) is \( G_{(p)}/G^{p^\alpha}_{(p)} \). But \( p \)-localization is an exact functor (see theorem 1.3.10), so the sequence

\[ 1 \to G^{p^\alpha}_{(p)} \to G_{(p)} \to (G/G^{p^\alpha})_{(p)} \to 1 \]

is exact, and the assertion follows. \( \square \)
Remark. We have shown that in order to obtain \( (G(p))_p \) it suffices to \( p \)-localize the inverse system corresponding to \( G \) (groups and projections maps) and then find its limit.

**Corollary 1.3.16.** The localization genus of any nilpotent group is contained in its completion genus.

Pickel proved that the completion genus of a finitely generated nilpotent group is finite ([14], [12]). Our corollary now yields a localization version of Pickel’s result:

**Theorem 1.3.17.** The localization genus of any finitely generated nilpotent group is finite.
Chapter 2

Localization of the $IA$-group of finitely generated, torsion-free, nilpotent, and metabelian groups

2.1 Powers of $IA$-automorphisms of torsion-free, metabelian, $p$-local, and nilpotent groups of class $c$

Let $G$ be the $p$-localization of a finitely generated, torsion-free, metabelian, and nilpotent group of class $c$.

Let $X = \{x_1, \ldots, x_r\}$ be a finite set that generates $G$ as a $p$-local group. Then $G$ is also generated, as a $p$-local group, by the set of basic commutators.
\[ B = \{b_1, \ldots, b_m\} \text{ on } X; \text{ where } b_i = x_i \text{ for } i = 1, \ldots, r. \] Since \( G \) is metabelian, any basic commutator on \( B \) is of the form

\[ \ldots[[[x_{i_3}, x_{i_2}], x_{i_1}], x_{i_4}], \ldots], x_{i_t}]. \]

Denote the weight of \( b_i \) as \( \text{wt}(b_i) \).

Let \( \varphi \in IA(G) \) and put

\[ \varphi(b_i) = b_i A_i \]

where \( \text{wt}(b_i) \leq c - 1 \) and \( A_i \in G' \). \( A_i \) can be expressed as a product of rational powers of basic commutators of weight at least 2, and at most \( c \). So we can write

\[ A_i = \prod_{k > l} [b_k, x_l]^{\frac{v(i)}{v(i) - 1}} \]

where \( \text{wt}(b_k) \leq c - 1 \) for each \( k \); and \( v(i) \) is relatively prime to \( p \).

For each \( k > l \), standard commutator calculus in this metabelian group gives:

\[ \varphi([b_k, x_l]) = [b_k, x_l][b_k, A_l][A_k, x_l]. \]

In order to find an expression for \( \varphi^m([b_k, x_l]) \), we do as follows:

\[ \varphi^2([b_k, x_l]) = \varphi([b_k, x_l])\varphi([b_k, A_l][A_k, x_l]) \]

\[ = [b_k, x_l][b_k, A_l][A_k, x_l][b_kA_k, \varphi(A_l)][\varphi(A_k), x_lA_l] \]

\[ = [b_k, x_l][b_k, A_l][A_k, x_l][b_k, \varphi(A_l)][\varphi(A_k), x_l]. \]

Continuing in this way we obtain:

\[ \varphi^m([b_k, x_l]) = [b_k, x_l] \prod_{i=0}^{m-1} [b_k, \varphi^i(A_l)][\varphi^i(A_k), x_l]. \quad (2.1) \]
We now wish to obtain a closed formula for \( \varphi^m([b_k, x_l]) \), one that does not depend on lower powers of \( \varphi \). The following computation gives this:

\[
\varphi(A_i) = \prod_{k > l} \varphi([b_k, x_l]) \frac{i_{kl}}{v(i)}
\]

\[
= \prod_{k > l} ([b_k, x_l][b_k, A_l][A_k, x_l]) \frac{i_{kl}}{v(i)}
\]

\[
= \prod_{k > l} [b_k, x_l] \frac{i_{kl}}{v(i)} \prod_{k > l} ([b_k, A_l][A_k, x_l]) \frac{i_{kl}}{v(i)}
\]

\[
= A_i \prod_{k > l} ([b_k, A_l][A_k, x_l]) \frac{i_{kl}}{v(i)}.
\]

Let

\[
\delta_{i1} = \prod_{k > l} ([b_k, A_l][A_k, x_l]) \frac{i_{kl}}{v(i)}.
\]

Then

\[
\varphi(A_i) = A_i \delta_{i1}.
\]

Next,

\[
\varphi(\delta_{i1}) = \prod_{k > l} ([\varphi(b_k), \varphi(A_l)][\varphi(A_k), \varphi(x_l)]) \frac{i_{kl}}{v(i)}
\]

\[
= \prod_{k > l} ([b_k A_k, A_l \delta_{i1}][A_k \delta_{k1}, x_l A_l]) \frac{i_{kl}}{v(i)}
\]

\[
= \prod_{k > l} ([b_k, A_l][b_k, \delta_{i1}][A_k, x_l][\delta_{k1}, x_l]) \frac{i_{kl}}{v(i)}
\]

\[
= \prod_{k > l} ([b_k, A_l][A_k, x_l]) \frac{i_{kl}}{v(i)} \prod_{k > l} ([b_k, \delta_{i1}][\delta_{k1}, x_l]) \frac{i_{kl}}{v(i)};
\]

so that

\[
\varphi(\delta_{i1}) = \delta_{i1} \delta_{i2}
\]

where

\[
\delta_{i2} = \prod_{k > l} ([b_k, \delta_{i1}][\delta_{k1}, x_l]) \frac{i_{kl}}{v(i)}.
\]
In general, for each $i$, we may construct a sequence of elements of $G'$:

$$A_i, \delta_{i1}, \delta_{i2}, \ldots$$ (2.2)

where

$$\delta_{i1} = \prod_{k > l} ([b_k, A_l][A_k, x_l])^{i_{kl}},$$

$$\delta_{ij} = \prod_{k > l} ([b_k, \delta_{l(j-1)}][\delta_{k(j-1)}, x_l])^{i_{kl}} \text{ for } j > 1,$$

$\varphi(A_i) = A_i \delta_{i1}$, and $\varphi(\delta_{ij}) = \delta_{ij} \delta_{i(j+1)}$.

Suppose that $A_i \in \gamma_z G$ for some integer $z$. A typical basic commutator $[b_k, x_l]$ appearing as a factor in $A_i$ has weight at least $z$, so that $b_k$ has weight at least $z - 1$ and $A_k \in \gamma_z$. This means that $[b_k, A_l][A_k, x_l] \in \gamma_{z+1}$, so $\delta_{i1} \in \gamma_{z+1}$. By making similar observations about higher terms of sequence 2.2, we see that $\delta_{ij} \in \gamma_{z+j}$.

We henceforth refer to sequence 2.2 as the $\delta$-sequence associated to $A_i$.

Consider the following computations:

- $\varphi(A_i) = A_i \delta_{i1}$
- $\varphi^2(A_i) = \varphi(A_i) \varphi(\delta_{i1}) = A_i \delta_{i1} \delta_{i1} \delta_{i2} = A_i \delta_{i1}^2 \delta_{i2}$
- $\varphi^3(A_i) = A_i \delta_{i1} (\delta_{i1} \delta_{i2})^2 \delta_{i2} \delta_{i3} = A_i \delta_{i1}^3 \delta_{i1}^2 \delta_{i2} \delta_{i3}$
- $\varphi^4(A_i) = A_i \delta_{i1} (\delta_{i1} \delta_{i2})^3 (\delta_{i2} \delta_{i3})^2 \delta_{i3} \delta_{i4} = A_i \delta_{i1}^4 \delta_{i2}^2 \delta_{i3}^3 \delta_{i4}$
- $\varphi^5(A_i) = A_i \delta_{i1} (\delta_{i1} \delta_{i2})^4 (\delta_{i2} \delta_{i3})^3 (\delta_{i3} \delta_{i4})^2 \delta_{i4} \delta_{i5} = A_i \delta_{i1}^5 \delta_{i2}^3 \delta_{i3}^2 \delta_{i4}^2 \delta_{i5}$

In general, we have
Lemma 2.1.1.

\[ \varphi^m(A_i) = A_i \delta_{i1}^{c_1(m)} \delta_{i2}^{c_2(m)} \cdots \delta_{im}^{c_m(m)} \]

where the numbers \( c_1(m), \ldots, c_m(m) \) correspond to to the \( m \)-th row of Pascal’s triangle (ignoring the left-most diagonal of 1’s):

\[
\begin{align*}
1 & \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
& \\
& \\
& \\
&
\end{align*}
\]

More precisely: \( c_j(m) = c_{j-1}(m-1) + c_j(m-1) \) if \( j > 1 \) and \( c_1(m) = m \).
Thus, \( c_j(m) \) is simply the binomial coefficient

\[ c_j(m) = \binom{m}{j} = \frac{m!}{n!(m-n)!}. \]

To find a formula for \( \varphi^m(b_i) \) we proceed as follows:

\[
A_i \varphi(A_i) \varphi^2(A_i) \cdots \varphi^{m-1}(A_i) = \\
A_i \left( A_i \delta_{i1}^{c_1(1)} \right) \left( A_i \delta_{i1}^{c_1(2)} \delta_{i2}^{c_2(2)} \right) \cdots \left( A_i \delta_{i1}^{c_1(m-1)} \delta_{i2}^{c_2(m-1)} \cdots \delta_{im}^{c_m(m-1)} \right) = \\
A_i^{m} \delta_{i1}^{c_1(1)+\cdots+c_1(m-1)} \delta_{i2}^{c_2(2)+\cdots+c_2(m-1)} \cdots \delta_{im}^{c_m(m-1)(m-1)}. \tag{2.3}
\]

Now put

\[ d_j(m) = c_j(j) + \cdots + c_j(m-1). \]

The following simple lemma will be extremely useful.
Lemma 2.1.2. 

\[ d_j(m) = \binom{m}{j+1}. \] 

(2.4)

Proof. This follows from direct calculation:

\[
\binom{m}{j+1} = \binom{m-1}{j} + \binom{m-1}{j+1} = \\
\binom{m-1}{j} + \binom{m-2}{j} + \binom{m-2}{j+1} = \\
\binom{m-1}{j} + \binom{m-2}{j} + \binom{m-3}{j} + \binom{m-3}{j+1} = \\
\binom{m-1}{j} + \binom{m-2}{j} + \binom{m-3}{j} + \cdots + \binom{j}{j} = d_j(m)
\]

since \( \binom{j}{j+1} = 0 \).

In particular

\[ d_1(m) = c_1(1) + \cdots + c_1(m-1) = 1 + 2 + \cdots + m - 1 = \frac{m(m-1)}{2}. \]

Rewriting equation 2.3 we obtain:

\[ A_i \varphi(A_i) \varphi^2(A_i) \cdots \varphi^{m-1}(A_i) = A_i \delta_i^{d_1(m)} \delta_i^{d_2(m)} \cdots \delta_i^{d_{m-1}(m)}. \]

Since the \( d_j(m) \) depend only on \( m \), and not on \( i \), we have proved:

Theorem 2.1.3. If

\[ \varphi(b_i) = b_i A_i, \]

then

\[ \varphi^m(b_i) = b_i A_i^m \delta_i^{d_1(m)} \delta_i^{d_2(m)} \cdots \delta_i^{d_{m-1}(m)}, \]

where

\[ d_j(m) = \binom{m}{j+1}. \]
2.2 Proof that $IA(G)$ is $p$-local

Consider the map

$$IA(G) \rightarrow IA(G)$$

$$\varphi \mapsto \varphi^n$$

where $(n, p) = 1$, and $G$ is as in section 2.1.

There are two results in this section. We begin with

**Theorem 2.2.1.**

$$IA(G) \rightarrow IA(G)$$

is one-to-one.

**Proof.** To see this, let

$$\varphi^n = \psi^n.$$  

We wish to prove that $\varphi = \psi$. For this purpose put

$$\varphi(b_i) = b_i A_i,$$

$$\psi(b_i) = b_i \hat{A}_i$$

where $1 \leq wt(b_i) \leq c - 1$.

As usual we write

$$A_i = \prod_{k > l} [b_k, x_l]^{i_{kl}}$$ \hspace{1cm} (2.5)

and

$$\hat{A}_i = \prod_{k > l} [b_k, x_l]^{\hat{i}_{kl}}.$$ \hspace{1cm} (2.6)

In order to show that $\varphi = \psi$ we proceed by reverse induction on $wt(b_i)$:
Suppose $wt(b_i) = c - 1$. Then both $A_i$ and $\hat{A}_i$ belong to $\gamma_c$. $\varphi^n(b_i) = \psi^n(b_i)$ implies that $b_iA_i^n = b_i\hat{A}_i^n$, so that $A_i^n = \hat{A}_i^n$. By $p$-locality this means that $A_i = \hat{A}_i$.

Now suppose that $b_i$ satisfies

$$1 \leq j = wt(b_i) \leq c - 2.$$ 

Assume the induction hypothesis that $\varphi = \psi$ on $\gamma_{j+1}$. Our goal is to show that $\varphi(b_i) = \psi(b_i)$. Since $\varphi^n(b_i) = \psi^n(b_i)$ then

$$b_iA_i^n \delta^{d_1}_i \cdots \delta^{d_{n-1}}_i = b_i\hat{A}_i^n \hat{\delta}^{d_1}_i \cdots \hat{\delta}^{d_{n-1}}_i.$$ 

Using the fact that we are in a $p$-local group, we obtain:

$$A_i \hat{A}_i^{-1} = (\delta^{d_1}_i \delta^{d_{n-1}}_i)^{\frac{d_1(n)}{n}} (\delta^{d_1}_i \delta^{d_{n-1}}_i)^{\frac{d_{n-1}(n)}{n}}. \quad (2.7)$$

Since $wt(b_i) = j$, $A_i$ and $\hat{A}_i$ each belongs to $\gamma_{j+1}$. In fact, by equation 2.7, the product $A_i \hat{A}_i^{-1}$ actually lies in $\gamma_{j+2}$. By induction:

$$\varphi(\hat{A}_i) = \psi(\hat{A}_i) = \hat{A}_i \delta^{d_1}_i.$$ 

This means that

$$\varphi(A_i \hat{A}_i^{-1}) = A_i \delta^{d_1}_i (\hat{A}_i \delta^{d_1}_i)^{-1} = A_i \hat{A}_i^{-1} \delta^{d_1}_i.$$ 

Simply because $\varphi$ is an IA-automorphism, it follows that

$$\delta^{d_1}_i \delta^{d_1}_i \in \gamma_{j+3}.$$ 

Similarly, if we evaluate $\varphi$ on any $\delta^{d_1}_m \hat{\delta}^{d_1}_m$ in the right hand side of 2.7 we have (again by induction on $j$) that

$$\varphi(\delta^{d_1}_m \hat{\delta}^{d_1}_m) = \delta^{d_1}_m \delta^{d_1}_m \delta^{d_1}_m \delta^{d_1}_m = (\delta^{d_1}_m \hat{\delta}^{d_1}_m \delta^{d_1}_m \hat{\delta}^{d_1}_m)^{-1}. \quad (2.8)$$
since
\[ \varphi(\hat{\delta}_{im}) = \psi(\hat{\delta}_{im}). \]
Again, because \( \varphi \) is and \( IA \)-automorphism, equation 2.8 implies that \( \delta_{i(m+1)} \hat{\delta}_{i(m+1)}^{-1} \) lies in a higher commutator subgroup than the commutator subgroup to which \( \delta_{im} \hat{\delta}_{im}^{-1} \) belongs. Hence the entire right-hand side of equation 2.7 belongs to \( \gamma_{j+3} \). We have established that
\[ A_i \hat{A}_i^{-1} \in \gamma_{j+3}. \]
Apply the same argument repeatedly to finally conclude that
\[ A_i \hat{A}_i^{-1} \in \gamma_{c+1} = 1. \]
This completes the proof that \( IA(G) \to IA(G) \) is one-to-one.

The second result of this section is that \( IA(G) \to IA(G) \) is onto. To see this we prove

**Theorem 2.2.2.** Let \( \{b_1, b_2, \ldots\} \) be the basic commutators on \( X = \{x_1, \ldots, x_r\} \) of weight at most \( c - 1 \). Let \( \varphi(b_i) = b_i A_i \in IA(G) \). Then, there exists an \( IA \)-automorphism \( \psi \) such that \( \psi^n = \varphi \).

**Proof.** **Part I.** Let \( \delta_{i1}, \delta_{i2}, \ldots, \delta_{i(c-2)} \) be the \( \delta \)-sequence associated to \( A_i \).\(^1\) We claim that it is possible to find \( p \)-local integers \( \epsilon_1(n), \ldots, \epsilon_{c-2}(n) \) that depend on \( n \) (and \( c \)), such that

- \( \psi(b_i) = b_i A_i^{1_{\epsilon_1(n)}} \delta_{i1}^{\epsilon_1(n)} \cdots \delta_{i(c-2)}^{\epsilon_{c-2}(n)} \in IA(G) \), and
- \( \psi^n = \varphi \).

\(^1\)Since \( G \) has class \( c \), we can henceforth assume that \( \delta_{il} = 1 \) for \( l > c - 2 \).
Construction of $\psi$:

Put

$$A_i = \prod_{k > l} [b_k, x_l]^{i_{kl}},$$

where the $i_{kl}$ are $p$-local integers.

We wish to find $p$-local integers $\epsilon_1, \ldots, \epsilon_{c-2}$, depending on $n$ and $c$ alone, such that

$$\psi(b_i) = b_i \alpha_i$$

gives an $IA$-automorphism where $\alpha_i = A_i^{\frac{1}{n}} \delta_{i1}^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-2}}$ and $\psi^n = \varphi$. We will show that these $\epsilon$'s can in fact be found by “solving” the equation $\psi^n = \varphi$.

Associated with $\alpha_i$ we have the corresponding sequence of deltas: $\hat{\delta}_{i1}, \ldots, \hat{\delta}_{i(c-2)}$.

To relate the $\hat{\delta}$’s with the $\delta$’s we do as follows:

$$\psi(A_i) = \prod_{k > l} [\psi(b_k), \psi(x_l)]^{i_{kl}} = \prod_{k > l} \left[ b_k A_i^{\frac{1}{n}} \delta_{i1}^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-2}}, x_l A_i^{\frac{1}{n}} \delta_{i1}^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-2}} \right]^{i_{kl}}.$$

Using commutator calculus in this metabelian group, and recalling the definitions of $A_i, \delta_{i1}, \ldots, \delta_{i(c-2)}$ we see that

$$\psi(A_i) = A_i \delta_{i1} \delta_{i2} \cdots \delta_{i(c-1)} \delta_{i(c-2)},$$

where $\delta_{i(c-1)} = 1$.

Analogous calculations give:

$$\psi(b_i) = b_i A_i^{\frac{1}{n}} \delta_{i1}^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-2}}$$

$$\psi(A_i) = A_i \delta_{i1}^{\frac{1}{n}} \delta_{i2}^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-3}}$$

$$\psi(\delta_{i1}) = \delta_{i1} \delta_{i2}^{\frac{1}{n}} \delta_{i3}^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-4}}$$

$$\psi(\delta_{i2}) = \delta_{i2} \delta_{i3}^{\frac{1}{n}} \delta_{i4}^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-5}}$$

$$\vdots$$

$$\psi(\delta_{i(c-2)}) = \delta_{i(c-2)}.$$
Next, we find the $\hat{\delta}_{ij}$:

$$
\psi(\alpha_i) = \psi\left(A\frac{1}{n} \delta_{i1}^{\alpha_{i1}} \cdots \delta_{i(c-2)}^{\alpha_{i(c-2)}}\right) = 
\left(A\frac{1}{n} \delta_{i1}^{\epsilon_{i1}} \delta_{i2}^{\epsilon_{i2}} \cdots \delta_{i(c-2)}^{\epsilon_{i(c-2)}}\right) 
\left(\delta_{i1}^{\epsilon_{1}^{c-1}} \delta_{i2}^{\epsilon_{2}^{c-1}} \cdots \delta_{i(c-3)}^{\epsilon_{c-2}^{c-1}}\right) 
\left(\delta_{i2}^{\epsilon_{2}^{c-2}} \delta_{i3}^{\epsilon_{3}^{c-2}} \cdots \delta_{i(c-3)}^{\epsilon_{c-3}^{c-2}}\right) 
\cdots 
\left(\delta_{i(c-2)}^{\epsilon_{c-2}^{c-2}}\right)
$$

By rearranging the above expression:

$$
\psi(\alpha_i) = \left(A\frac{1}{n} \delta_{i1}^{\epsilon_{i1}} \delta_{i2}^{\epsilon_{i2}} \cdots \delta_{i(c-2)}^{\epsilon_{i(c-2)}}\right) 
\left(\delta_{i1}^{\epsilon_{1}^{c-1}} \delta_{i2}^{\epsilon_{2}^{c-1}} \cdots \delta_{i(c-3)}^{\epsilon_{c-2}^{c-1}}\right) 
\left(\delta_{i2}^{\epsilon_{2}^{c-2}} \delta_{i3}^{\epsilon_{3}^{c-2}} \cdots \delta_{i(c-3)}^{\epsilon_{c-3}^{c-2}}\right) 
\cdots 
\left(\delta_{i(c-2)}^{\epsilon_{c-2}^{c-2}}\right)
$$

From this we see that:

$$
\hat{\delta}_{i1} = \left(\delta_{i1}^{\frac{1}{n}}\right) \left(\delta_{i2}^{\epsilon_{1}^{c-1} + \frac{1}{n} \epsilon_{1}}\right) \left(\delta_{i3}^{\epsilon_{2}^{c-2} + \epsilon_{1} \epsilon_{1} + \frac{1}{n} \epsilon_{2}}\right) \left(\delta_{i4}^{\epsilon_{3}^{c-3} + \epsilon_{2} \epsilon_{1} + \epsilon_{1} \epsilon_{2} + \frac{1}{n} \epsilon_{3}}\right) \cdots \left(\delta_{i(c-2)}^{\epsilon_{c-2}^{c-2} + \epsilon_{c-3} \epsilon_{1} + \cdots + \frac{1}{n} \epsilon_{c-3}}\right).
$$

This expresses $\hat{\delta}_{i1}$ in terms of the $\delta_{ij}$'s.

Rewriting the exponents in equation 2.9 we get

$$
\hat{\delta}_{i1} = \delta_{i1}^{\frac{1}{n} \epsilon_{1}} \delta_{i2}^{\epsilon_{1}^{c-1}} \delta_{i3}^{\epsilon_{2}^{c-2}} \cdots \delta_{i(c-2)}^{\epsilon_{c-2}^{c-2}}.
$$

(2.9)
where each $\alpha_j$ depends on $n, \epsilon_1, \ldots, \epsilon_j$. Symbolically:

$$\alpha_j = \alpha_j(n, \epsilon_1, \epsilon_2, \ldots, \epsilon_j)$$

for $j = 1, 2, \ldots, c - 3$.

In order to find an expression for $\hat{\delta}_{i2}$, evaluate $\psi$ on $\hat{\delta}_{i1}$:

$$\psi(\hat{\delta}_{i1}) = \psi\left(\delta_1^{\frac{1}{n_1}} \delta_2^{\alpha_1} \delta_3^{\alpha_2} \cdots \delta_{i(c-2)}^{\alpha_{c-3}}\right) =$$

$$= \left(\delta_1^{\frac{1}{n_1}} \delta_2^{\frac{1}{n_2}} \delta_3^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-4}}\right)^{\alpha_1} \cdots \left(\delta_{i(c-3)}^{\frac{1}{n(c-3)}} \delta_{i(c-2)}^{\epsilon_{c-3}}\right)^{\alpha_{c-3}} =$$

$$= \left(\delta_1^{\frac{1}{n_1}} \delta_2^{\frac{1}{n_2}} \delta_3^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-4}}\right) \left(\delta_2^{\frac{1}{n_2}} \delta_3^{\alpha_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-4}}\right) \cdots \left(\delta_{i(c-2)}^{\epsilon_{c-4}}\right).$$

Since the first term of the right-hand side is $\hat{\delta}_{i1}$, we readily obtain

$$\hat{\delta}_{i2} = \delta_2^{\frac{1}{n_2}} \delta_3^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-4}}.$$ 

Again, we rewrite the exponents in this equation to obtain:

$$\hat{\delta}_{i2} = \delta_2^{\frac{1}{n_2}} \delta_3^{\beta_1} \cdots \delta_{i(c-2)}^{\beta_{c-4}},$$

where

$$\beta_j = \beta_j(n, \epsilon_1, \ldots, \epsilon_j; \alpha_1, \ldots, \alpha_j).$$

Since $\alpha_j = \alpha_j(n, \epsilon_1, \ldots, \epsilon_j)$, we can in fact write:

$$\beta_j = \beta_j(n, \epsilon_1, \ldots, \epsilon_j)$$

where $j = 1, \ldots, c - 4$.

Continue computing the $\hat{\delta}_{ij}$’s in this way, to finally get

$$\hat{\delta}_{i(c-3)} = \delta_{i(c-3)}^{\frac{n}{2}} \delta_{i(c-2)}^{\epsilon_{c-2}}.$$
where $\tau_1 = \tau_1(n, \epsilon_1)$, and
\[ \hat{\delta}_{i(c-2)} = \delta_{i(c-2)}^{\left(\frac{1}{n}\right)^{c-1}}. \]

We now show that the $\epsilon$'s appearing in the equation
\[ \psi^n(b_i) = \varphi(b_i) \tag{2.10} \]
can be chosen so that they depend only on $n$. For this purpose we use
the formula that describes powers of $IA$-automorphisms (see theorem 2.1.3).

Equation 2.10 becomes:
\[ A_i \delta_{i1}^{n\epsilon_1} \cdots \delta_{i(c-2)}^{n\epsilon_{c-2}} \hat{\delta}_{i1} \cdots \hat{\delta}_{i(c-2)} = A_i. \]

Canceling $A_i$ and rewriting the $\hat{\delta}$'s in terms of the $\delta$'s yields:
\[ \delta_{i1}^{n\epsilon_1} \cdots \delta_{i(c-2)}^{n\epsilon_{c-2}} \left( \delta_{i1}^{\frac{1}{n}} \delta_{i2}^{\alpha_1} \cdots \delta_{i(c-2)}^{\alpha_{c-3}} \delta_{i1}^{\beta_1} \cdots \delta_{i(c-2)}^{\beta_{c-4}} \right)^{d_1} \cdots \left( \delta_{i(c-2)}^{\frac{1}{n^{c-1}}} \right)^{d_{c-2}} = 1. \]

Now solve for each $\epsilon_i$ in the following way:

- $n\epsilon_1 + \frac{1}{n^2}d_1 = 0 \Rightarrow \epsilon_1 = -\frac{1}{n^3}d_1$, which means that $\epsilon_1$ can be chosen so that it depends on $n$ only.

- $n\epsilon_2 + \alpha_1 d_1 + \frac{1}{n}d_2 = 0 \Rightarrow \epsilon_2 = -\frac{1}{n} \alpha_1 d_1 - \frac{1}{n^2}d_2$. Since $\alpha_1$ depends on $n$ and $\epsilon_1$, we conclude that $\epsilon_2$ can be chosen so that it depends on $n$ only.

- $n\epsilon_3 + \alpha_2 d_1 + \beta_1 d_2 + \frac{1}{n^3}d_3 = 0 \Rightarrow \epsilon_3 = -\frac{1}{n} \alpha_2 d_1 - \frac{1}{n} \beta_1 d_2 - \frac{1}{n^3}d_3$. Again, $\alpha_2$ depends on $n$, $\epsilon_1$, and $\epsilon_2$; so $\epsilon_3$ can be chosen as to depend on $n$ only.

Continue this process to choose $\epsilon_1, \ldots, \epsilon_{c-2}$ so that they depend on $n$ alone.

**Part II.** Consider the following map, defined on the “original” $p$-generators
\{ $x_1, \ldots, x_r$ \} of $G$:
\[ \psi(x_i) = x_i A_i^{\frac{1}{n}} \delta_{i1}^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-2}}. \]
Our first task is to show that $\psi$ can be extended to a well defined homomorphism

$$gp(x_1, \ldots, x_r) \rightarrow G$$

(and can therefore be “lifted” to a self-homomorphism of $G$).

For this purpose, suppose that $r$ is a trivial word in the $x$’s:

$$r = x_{t_1} \cdots x_{t_s} = 1.$$  

We need to show that

"$\psi(x_{t_1}) \cdots \psi(x_{t_s}) = 1$.”

A straightforward computation yields:

$$\psi(x_{t_1})\psi(x_{t_2})\cdots\psi(x_{t_s}) =$$

$$\left(x_{t_1} A_{t_1}^{\frac{1}{c-2}} \delta_{t_1} \cdots \delta_{t_1(c-2)}\right) \left(x_{t_2} A_{t_2}^{\frac{1}{c-2}} \delta_{t_2} \cdots \delta_{t_2(c-2)}\right) \cdots \left(x_{t_s} A_{t_s}^{\frac{1}{c-2}} \delta_{t_s} \cdots \delta_{t_s(c-2)}\right) =$$

$$x_{t_1} \cdots x_{t_s} (A_{t_1} \cdots A_{t_s})^{\frac{1}{c-2}} \delta_{t_1(c-2)} \cdots \delta_{t_s(c-2)} \prod_{k<l} [\omega_{t_k}, x_{t_l}],$$

where

$$\omega_{t_j} = A_{t_j}^{\frac{1}{c-2}} \delta_{t_j} \cdots \delta_{t_j(c-2)}.$$  

Remembering that $r = 1$, we have:

$$\psi(x_{t_1}) \cdots \psi(x_{t_s}) = (A_{t_1} \cdots A_{t_s})^{\frac{1}{c-2}} \delta_{t_1} \cdots \delta_{t_s} \cdots \delta_{t_s(c-2)} \prod_{k<l} [\omega_{t_k}, x_{t_l}] \cdot$$

(2.11)

On the other hand:

$$1 = \varphi(x_{t_1} \cdots x_{t_s}) = x_{t_1} A_{t_1} \cdots x_{t_s} A_{t_s} =$$

$$x_{t_1} \cdots x_{t_s} A_{t_1} \cdots A_{t_s} \prod_{k<l} [A_{t_k}, x_{t_l}].$$

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Again, since \( x_{t_1} \cdots x_{t_s} = 1 \) we find that

\[
1 = A_{t_1} \cdots A_{t_s} \prod_{k < l} [A_{t_k}, x_{t_l}].
\]  

(2.12)

Taking the \( n \)th power on both sides of equation 2.11 yields

\[
(\psi(x_{t_1}) \cdots \psi(x_{t_s}))^n =
\]

\[
(A_{t_1} \cdots A_{t_s}) (\delta_{t_1}^{(c-2)} \cdots \delta_{t_s}^{(c-2)}) \prod_{k < l} [\omega_{t_k}, x_{t_l}]^n =
\]

\[
(A_{t_1} \cdots A_{t_s}) (\delta_{t_1}^{(c-2)} \cdots \delta_{t_s}^{(c-2)}) \prod_{k < l} [\omega_{t_k}^{n}, x_{t_l}].
\]

(2.13)

Now, since

\[
\omega_{t_k} = A_{t_k}^{\frac{1}{n}} \delta_{t_k}^{(c-2)} \cdots \delta_{t_k}^{(c-2)}, \text{ then } \omega_{t_k}^n = A_{t_k} \delta_{t_k}^{(c-2)} \cdots \delta_{t_k}^{(c-2)},
\]

so

\[
\prod_{k < l} [\omega_{t_k}^{n}, x_{t_l}] = \prod_{k < l} [A_{t_k}, x_{t_l}] [\delta_{t_k}^{(c-2)}, x_{t_l}] \cdots [\delta_{t_k}^{(c-2)}, x_{t_l}] .
\]

(2.14)

Substituting 2.14 in 2.13, together with equation 2.12 and commutator calculus gives

\[
(\psi(x_{t_1}) \cdots \psi(x_{t_s}))^n =
\]

\[
(\delta_{t_1}^{(c-2)} \cdots \delta_{t_s}^{(c-2)}) \prod_{k < l} [\delta_{t_k}^{(c-2)}, x_{t_l}]^{c-2}.
\]

(2.15)

Since we are in a \( p \)-local group, we may take \( n \)th roots on both sides, so that

\[
\psi(x_{t_1}) \cdots \psi(x_{t_s}) =
\]

\[
(\delta_{t_1}^{(c-2)} \cdots \delta_{t_s}^{(c-2)})^{c-2} \prod_{k < l} [\delta_{t_k}^{(c-2)}, x_{t_l}]^{c-2} .
\]

(2.16)
Next, apply $\varphi$ to equation 2.12 to obtain

$$1 = A_{t_1} \delta_{t_1} \cdots A_{t_s} \delta_{t_s} \prod_{k<l} [A_{t_k} \delta_{t_k}, x_{t_l} A_{t_l}] = A_{t_1} \cdots A_{t_s} \prod_{k<l} [A_{t_k}, x_{t_l}] \delta_{t_1} \cdots \delta_{t_s} \prod_{k<l} [\delta_{t_k}, x_{t_l}].$$

Using equation 2.12 itself we conclude that

$$\delta_{t_1} \cdots \delta_{t_s} \prod_{k<l} [\delta_{t_k}, x_{t_l}] = 1. \quad (2.17)$$

Apply $\varphi$ to 2.17 to get

$$\delta_{t_1}^2 \cdots \delta_{t_s}^2 \prod_{k<l} [\delta_{t_k}, x_{t_l}] = 1. \quad (2.18)$$

We do this repeatedly to finally conclude (from equation 2.16) that

$$\psi(x_{t_1}) \cdots \psi(x_{t_s}) = 1,$$

as promised.

We have proved the following two major facts:

1. The map $\psi$ defined on the $p$-generators of $G$ as

$$\psi(x_i) = x_i A_{t_1}^{\frac{1}{n}} \delta_{t_1}^{\epsilon_1} \cdots \delta_{t(c-2)}^{\epsilon_{c-2}}$$

extends to a well-defined homomorphism from $gp(x_1, \ldots, x_r)$ to $G$, and can therefore be lifted to a self-homomorphism of $G$.

2. $\psi$ satisfies

$$x_i^{-1} \psi(x_i) \in G' = [G, G]$$

for all $i$, and

$$\psi^n = \varphi.$$
We now give a very simple argument to establish that $\psi$ is in fact an automorphism, and therefore an $IA$-automorphism. To show that $\psi$ is one-to-one, choose $g \in \ker \psi$. Then $\psi(g) = 1$, so that $\psi^n(g) = 1$. Since $\psi^n = \varphi$, $\varphi(g) = 1$. We conclude that $g \in \ker \varphi$. Since $\varphi$ is one-to-one, $g = 1$. Hence, $\psi$ is one-to-one.

Next choose $g \in G$. Since $\varphi$ is onto, there exists a $g' \in G$ so that $\varphi(g') = g$. Thus, $\psi^n(g') = g$, which implies that $\psi(\psi^{n-1}(g')) = g$. This means that $\psi$ is onto.

Theorem 2.2.2 is now proved. □

### 2.3 $IA(G) \rightarrow IA(G_{(p)})$ is a $p$-isomorphism

Let $G = gp(x_1, \ldots, x_m)$ be a finitely generated, torsion-free, metabelian, and nilpotent group of class $c$. Let $G_{(p)}$ be its localization at the prime $p$. Consider the localization diagram

\[
\begin{array}{ccc}
G & \rightarrow & G \\
\downarrow e & & \downarrow e \\
G_{(p)} & \rightarrow & G_{(p)}
\end{array}
\]

and observe that if $f \in IA(G)$, then $f_p \in IA(G_{(p)})$.

**Lemma 2.3.1.** The homomorphism

\[
IA(G) \rightarrow IA(G_{(p)})
\]

\[
f \mapsto f_p
\]

is a monomorphism (and hence $p$-injective).
Proof. Let \( f \in \ker (IA(G) \to IA(G(p))) \) and put \( f(x_i) = x_iA_i \) where \( A_i \) lies in \( G' = [G,G] \). For \( g \in G \), write \( e(g) = \bar{g} \). If \( x_i \) is a generator of \( G \), \( \bar{x}_i \) belongs to \( G(p) \) and

\[
\bar{x}_i = f_p(\bar{x}_i) = \overline{f(x_i)} = \bar{x}_iA_i.
\]

Hence, \( \bar{A}_i = \bar{1} \) in \( G'(p) \), which means that \( A_i \) belongs to \( \ker (e : G' \to G'(p)) \).

Since \( G' \) is torsion-free and \( e : G' \to G'(p) \) is a localization map, \( e \) is one-to-one so that \( A_i = 1 \). Hence \( f(x_i) = x_i \); and therefore

\[
IA(G) \to IA(G(p))
\]

\[
f \mapsto f_p
\]

is indeed a monomorphism. \( \square \)

Remark. The discussion that led to the \( \delta \)-sequence (see 2.2) and theorem 2.1.3 is also valid for \( IA(G) \), where \( G \) is finitely generated, torsion-free, nilpotent, and metabelian (the condition of \( G \) being \( p \)-local is dropped). This observation will be used in the sequel.

Lemma 2.3.2. \( IA(G) \to IA(G(p)) \) is \( p \)-surjective.

Proof. Let \( \varphi \in IA(G(p)) \). Consider the action of \( \varphi \) on the “\( p \)-generators” of \( G(p) \): \( \varphi(\bar{x}_i) = \bar{x}_iA_i \), where \( A_i \in G'(p) \). Since \( e : G' \to G'(p) \) is \( p \)-surjective, there exists and integer \( s_i \), relatively prime to \( p \), such that \( A_i^{s_i} \) belongs to the image of \( e : G' \to G'(p) \) for each \( i = 1, 2, \ldots, m \). Put

\[
\sigma_1 = s_1s_2 \ldots s_m.
\]

\( A_i^{\sigma_1} \) clearly lies in the image of \( e : G' \to G'(p) \) (for each \( i = 1, 2, \ldots, m \)) because such image is a subgroup of \( G'(p) \). Similarly, choose \( \sigma_2, \ldots, \sigma_{c-1} \) (independent
if \( i \) so that \( \delta_{ik}^{\sigma_{k+1}} \) belongs to the image of \( e : \gamma_{k+2}(G) \to \gamma_{k+2}(G(p)) \), for \( k = 1, \ldots, c - 2 \). Let

\[
\sigma = \sigma_1 \sigma_2 \cdots \sigma_{c-1}.
\]

We have the following:

1. \( \sigma \) is relatively prime to \( p \),

2. \( \sigma^e \) belongs to the image of \( e : G' \to G'_p \), and

3. \( \delta_{ik}^{\sigma_{k+1}} \) belongs to the image of \( e : \gamma_{k+2}(G) \to \gamma_{k+2}(G(p)) \), \( (k = 1, \ldots, c - 2) \).

Using lemma 2.1.2, we see that

- \( d_1(\sigma) = \binom{\sigma}{2} = \frac{\sigma(\sigma-1)}{2} \)
- \( d_2(\sigma) = \binom{\sigma}{3} = \frac{\sigma(\sigma-1)(\sigma-2)}{3!} \)
- \( d_3(\sigma) = \binom{\sigma}{4} = \frac{\sigma(\sigma-1)(\sigma-2)(\sigma-3)}{4!} \)
- \( \vdots \)
- \( d_{c-2}(\sigma) = \binom{\sigma}{c-1} = \frac{\sigma(\sigma-1)(\sigma-2)\cdots(\sigma-c+2)}{(c-1)!} \).

Fix \( 1 \leq j \leq c - 2 \) and consider the number

\[
d_j(\sigma) = \binom{\sigma}{j+1} = \frac{\sigma(\sigma-1)(\sigma-2)\cdots(\sigma-j)}{(j+1)!}.
\]

Write

\[(j + 1)! = p^{\alpha_j} \epsilon_j\]

where \( p \) and \( \epsilon_j \) are relatively prime. (If \( p \) does not divide \((j+1)! \) take \( \alpha_j = 0 \).)

Next, let

\[
\epsilon = \epsilon_1 \epsilon_2 \cdots \epsilon_{c-2}.
\]
Notice that $p$ and $\epsilon$ are relatively prime. Now put

$$s = \epsilon \sigma.$$  

Again, $p$ and $s$ are relatively prime. Invoking lemma 2.1.2 once more, we see that for each $j$:

$$d_j(s) = \frac{\epsilon_1 \epsilon_2 \cdots \epsilon_{c-2} \sigma (s - 1)(s - 2) \cdots (s - j)}{p^{\alpha_j} \epsilon_j} = \frac{\epsilon_1 \epsilon_2 \cdots \epsilon_{j-1} \epsilon_{j+1} \cdots \epsilon_{c-2} \sigma (s - 1)(s - 2) \cdots (s - j)}{p^{\alpha_j}}.$$  

As $d_j(s)$ is an integer, and $p^{\alpha_j}$ does not divide $\epsilon_1 \epsilon_2 \cdots \epsilon_{j-1} \epsilon_{j+1} \cdots \epsilon_{c-2} \sigma$, $p^{\alpha_j}$ has to divide $(s-1)(s-2) \cdots (s-j)$. Hence, each integer $s, d_1(s), d_2(s), \ldots, d_{c-2}(s)$ is a multiple of $\sigma$. The crucial conclusion is that

1. $s$ is relatively prime to $p$,

2. $A_i^s$ belongs to the image of $e : G' \rightarrow G'_{(p)}$, and

3. $\delta_{ij}^{d_j(s)}$ lies in the image of $e : \gamma_{j+2}(G) \rightarrow \gamma_{j+2}(G_{(p)})$, for $j = 1, 2, \ldots, c-2$.

We can therefore choose $\alpha_i \in G'$ such that $\overline{\alpha}_i = A_i^s$, and $D_{ij} \in \gamma_{j+2}(G)$ such that $\overline{D}_{ij} = \delta_{ij}^{d_j(s)}$ for each $j = 1, 2, \ldots, c-2$; where $\overline{g} = e(g)$ for $g \in G$.

Using theorem 2.1.3 we see that

$$\varphi^s(\overline{x}_i) = \overline{x}_i A_i^s \delta_{i1}^{d_1(s)} \cdots \delta_{i(c-2)}^{d_{c-2}(s)}.$$  

Let

$$\beta_i = \alpha_i D_{i1} \cdots D_{i(c-2)} \in G'.$$

Define the following map on the generators of $G$:

$$f(x_i) = x_i \beta_i.$$  

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If we can show that $f$ can be extended to an element of $IA(G)$, then $f_p$ and $\varphi^*$ will coincide on the $p$-generators of $G_p$ and will therefore be equal as $IA$-automorphisms.

Using similar techniques as before, we prove first that $f$ extends to a self-homomorphism of $G$. For this purpose let $r = x_{t_1}x_{t_2} \cdots x_{t_v}$ be a trivial word in the generators of $G$. Then

$$f(x_{t_1}) \cdots f(x_{t_v}) = (x_{t_1}\beta_{t_1}) \cdots (x_{t_v}\beta_{t_v}) =$$

$$(x_{t_1} \cdots x_{t_v})(\beta_{t_1} \cdots \beta_{t_v}) \prod_{l<m} [\beta_{t_l}, x_{t_m}] =$$

$$(\beta_{t_1} \cdots \beta_{t_v}) \prod_{l<m} [\beta_{t_l}, x_{t_m}].$$

We thus have:

$$f(x_{t_1}) \cdots f(x_{t_v}) = (\beta_{t_1} \cdots \beta_{t_v}) \prod_{l<m} [\beta_{t_l}, x_{t_m}]. \tag{2.19}$$

Next we work with $\varphi$. Since $r = x_{t_1} \cdots x_{t_v} = 1$, then

$$\overline{r} = \bar{x}_{t_1} \bar{x}_{t_2} \cdots \bar{x}_{t_v} = \bar{1}.$$ 

Thus

$$\bar{1} = \varphi(\overline{r}) = (\overline{x_{t_1}}A_{t_1}) \cdots (\overline{x_{t_v}}A_{t_v}) =$$

$$\overline{x_{t_1}} \cdots \overline{x_{t_v}} A_{t_1} \cdots A_{t_v} \prod_{l<m} [A_{t_l}, \overline{x_{t_m}}] =$$

$$A_{t_1} \cdots A_{t_v} \prod_{l<m} [A_{t_l}, \overline{x_{t_m}}].$$

Hence

$$\bar{1} = A_{t_1} \cdots A_{t_v} \prod_{l<m} [A_{t_l}, \overline{x_{t_m}}]. \tag{2.20}$$
Apply $\varphi$ to both sides of equation 2.20 to get

$$
\mathbb{1} = (A_{t_1} \delta_{t_1}) \cdots (A_{t_v} \delta_{t_v}) \prod_{l<m} [A_{t_l} \delta_{t_l}, x_{t_m} A_{t_m}] = 
A_{t_1} \cdots A_{t_v} \prod_{l<m} [A_{t_l}, x_{t_m}] \delta_{t_1} \cdots \delta_{t_v} \prod_{l<m} [\delta_{t_l}, x_{t_m}] = \mathbb{1}.
$$

We conclude that

$$
\delta_{t_1} \cdots \delta_{t_v} \prod_{l<m} [\delta_{t_l}, x_{t_m}] = \mathbb{1}.
$$

Iterate this process to obtain

$$
\delta_{t_{1j}} \cdots \delta_{t_{vj}} \prod_{l<m} [\delta_{t_{lj}}, x_{t_{jm}}] = \mathbb{1} \quad (2.21)
$$

for all $j = 1, 2, \ldots, c - 2$.

By equation 2.19 and the definition of $\mathcal{B}_i$:

$$
f(x_{t_1}) \cdots f(x_{t_v}) \quad = \quad
(A_{t_1} \cdots A_{t_v})^s (\delta_{t_1} \cdots \delta_{t_v})^{d_1} \cdots (\delta_{t_1(c-2)} \cdots \delta_{t_v(c-2)})^{d_{c-2}} \prod_{l<m} [A_{t_l}^s \delta_{t_l}^{d_1} \cdots \delta_{t_l(c-2)}^{d_{c-2}}, x_{t_m}] =

(A_{t_1} \cdots A_{t_v})^s (\delta_{t_1} \cdots \delta_{t_v})^{d_1} \cdots (\delta_{t_1(c-2)} \cdots \delta_{t_v(c-2)})^{d_{c-2}} \prod_{l<m} \left( [A_{t_l}, x_{t_m}]^s [\delta_{t_l1}, x_{t_m}]^{d_1} \cdots [\delta_{t_l(c-2)}, x_{t_m}]^{d_{c-2}} \right) =

(A_{t_1} \cdots A_{t_v} \prod_{l<m} [A_{t_l}, x_{t_m}])^s \left( \delta_{t_1} \cdots \delta_{t_v} \prod_{l<m} [\delta_{t_l1}, x_{t_m}] \right)^{d_1} \cdots \left( \delta_{t_1(c-2)} \cdots \delta_{t_v(c-2)} \prod_{l<m} [\delta_{l(c-2)}, x_{t_m}] \right)^{d_{c-2}} = \mathbb{1}
$$

(by equations 2.20 and 2.21). We have shown that

$$
\overline{f(x_{t_1}) \cdots f(x_{t_v})} = \mathbb{1};
$$

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in other words
\[ f(x_{t_1}) \cdots f(x_{t_v}) \in \ker \left( e : G \to G(p) \right). \]

As \( G \) is torsion-free, \( e \) is in fact one-to-one; so that
\[ f(x_{t_1}) \cdots f(x_{t_v}) = 1. \]

\( f \) thus extends to a self-homomorphism of \( G \).

Besides proving that \( f \) extends to a self-homomorphism of \( G \), we have seen (by construction) that
\[ g^{-1} f(g) \in G' \]
for all \( g \in G \) and
\[ f_p = \varphi^s \]
on the \( p \)-generators of \( G(p) \) (and hence on all of \( G(p) \)).

We now provide a very simple argument to show that \( f \) is an \( IA \)-automorphism.
Contemplate the localization diagram:

\[
\begin{array}{ccc}
G & \xrightarrow{f} & G \\
\downarrow e & & \downarrow e \\
G(p) & \xrightarrow{f_p} & G(p).
\end{array}
\]

Since \( f_p \) is already an \( IA \)-automorphism and \( G \) is torsion-free, both \( e \) and \( f_p \) are one-to-one. Choose \( g \in \ker(f) \). Then, by commutativity of the diagram, \( f_p(e(g)) = 1 \). It follows readily that \( g = 1 \) and \( f \) is one-to-one.

To prove that \( f \) is onto we argue as follows: using commutator calculus one can prove that if \( g_j \in \gamma_j \) then \( g_j^{-1} f(g_j) \in \gamma_{j+1} \). In particular, \( f \) is the identity on \( \gamma_c \) and hence (trivially) onto there. We now do reverse induction.
on $j$. Suppose $f$ is onto on $\gamma_{j+1}$. We show that “$f$ is onto on $\gamma_j$”: let $g_j \in \gamma_j$, then, by induction

$$f(g_j) = g_jg_{j+1} = g_jf(h_{j+1})$$

for some $g_{j+1}, h_{j+1} \in \gamma_{j+1}$. Hence

$$f(g_jh_{j+1}^{-1}) = g_j$$

where clearly $g_jh_{j+1}^{-1} \in \gamma_j$. Thus $f$ is onto on $\gamma_j$. This completes the proof of the lemma.

The main result of this thesis follows, which we state as

**Theorem 2.3.3.** Let $G$ be finitely generated, metabelian, and nilpotent of class $c$; and $G_{(p)}$ its localization at the prime $p$. Then, the natural map

$$IA(G) \to IA(G_{(p)})$$

is a $p$-isomorphism.

If it were the case that $InnG = IA(G)$ for all nilpotent groups $G$, theorem 2.3.3 would be trivial since we would have

$$IA(G)_{(p)} \cong (G/\zeta)_{(p)} \cong G_{(p)}/\zeta_{(p)} \cong IA(G_{(p)}).$$

To demonstrate that our theorem is nontrivial in general we compute $InnG$ and $IA(G)$ where $G$ is free nilpotent of class 2 and rank 3. It will then be clear that $InnG \neq IA(G)$.

Let

$$G = \langle x, y, z \rangle$$
be free nilpotent of class 2 on the generators \( \{x, y, z\} \). Put \( c_{12} = [x, y] \), \( c_{13} = [x, z] \), and \( c_{23} = [y, z] \). Then

\[
\{x, y, z, c_{12}, c_{13}, c_{23}\}
\]

is a basic sequence of basic commutators on \( \{x, y, z\} \). Since \( G \) is free nilpotent, every \( g \in G \) can be uniquely written as

\[
g = x^{e_1}y^{e_2}z^{e_3}c_{12}^{e_{12}}c_{13}^{e_{13}}c_{23}^{e_{23}}.
\]

If \( g' = x^{e'_1}y^{e'_2}z^{e'_3}c_{12}^{e'_{12}}c_{13}^{e'_{13}}c_{23}^{e'_{23}} \) is another element of \( G \), standard commutator calculus gives

\[
gg' = x^{e_1+e'_1}y^{e_2+e'_2}z^{e_3+e'_3}c_{12}^{e_{12}+e'_{12}}c_{13}^{e_{13}+e'_{13}}c_{23}^{e_{23}+e'_{23}} - e_1^{e_2} - e_2^{e_1} - e_3^{e_2} - e_2^{e_1}.
\] (2.22)

Consider the following nine elements of \( IA(G) \):

- \( \varphi_1(x) = xc_{12}, \varphi_1(y) = y, \varphi_1(z) = z; \varphi_2(x) = x, \varphi_2(y) = yc_{12}, \varphi_2(z) = z; \)
- \( \varphi_3(x) = x, \varphi_3(y) = y, \varphi_3(z) = zc_{12}; \varphi_4(x) = xc_{13}, \varphi_4(y) = y, \varphi_4(z) = z; \)
- \( \varphi_5(x) = x, \varphi_5(y) = yc_{13}, \varphi_5(z) = z; \varphi_6(x) = x, \varphi_6(y) = y, \varphi_6(z) = zc_{13}; \)
- \( \varphi_7(x) = xc_{23}, \varphi_7(y) = y, \varphi_7(z) = z; \varphi_8(x) = x, \varphi_8(y) = yc_{23}, \varphi_8(z) = z; \)

- \( \varphi_9(x) = x, \varphi_9(y) = y, \varphi_9(z) = zc_{23}. \)

For any \( \varphi \in IA(G) \) we can write

\[
\varphi(x) = x^{a_{12}}c_{12}^{c_{12}}c_{23}^{c_{23}},
\]

\[
\varphi(y) = y^{a_{12}}c_{12}^{c_{12}}c_{23}^{c_{23}},
\]

\[
\varphi(z) = z^{a_{12}}c_{12}^{c_{12}}c_{23}^{c_{23}}.
\]
It is straightforward to show that $\varphi$ is uniquely expressed as

$$\varphi = \varphi_1^{a_1} \varphi_2^{a_2} \varphi_3^{a_3} \varphi_4^{b_4} \varphi_5^{b_5} \varphi_6^{b_6} \varphi_7^{c_7} \varphi_8^{c_8} \varphi_9^{c_9}.$$ 

Since $IA(G)$ is a torsion-free abelian group, this proves that

$$IA(G)$$

is free abelian of rank 9. \hfill (2.23)

Now choose $\varphi \in InnG$. By definition of $InnG$, there exists $g = x^{e_1} y^{e_2} z^{e_3} c_{12}^{e_{12}} c_{13}^{e_{13}} c_{23}^{e_{23}} \in G$ such that

$$\varphi(x) = g^{-1}xg, \quad \varphi(y) = g^{-1}yg, \quad \varphi(z) = g^{-1}zg.$$ 

By the normal form 2.22 we readily find that

$$g^{-1} = x^{-e_1} y^{-e_2} z^{-e_3} c_{12}^{-e_{12}} c_{13}^{-e_{13}} c_{23}^{-e_{23}}.$$ \hfill (2.24)

and further use of 2.22 ultimately gives

$$\varphi(x) = xc_{12}^{e_{12}} c_{13}^{e_{13}}, \quad \varphi(y) = yc_{12}^{-1} c_{23}^{e_{23}}, \quad \varphi(z) =zc_{13}^{-1} c_{23}^{e_{23}}.$$ 

We obtain three specific elements of $InnG$ by setting, in turn, $e_1 = 1, e_2 = e_3 = 0; e_2 = 1, e_1 = e_3 = 0; \text{and} e_3 = 1, e_1 = e_2 = 0$. These are:

$$\varphi_1(x) = x, \quad \varphi_1(y) = yc_{12}^{-1}, \quad \varphi_1(z) = zc_{13}^{-1};$$

$$\varphi_2(x) = xc_{12}, \quad \varphi_2(y) = y, \quad \varphi_2(z) = zc_{23}^{-1};$$

$$\varphi_3(x) = xc_{13}, \quad \varphi_3(y) = yc_{23}, \quad \varphi_3(z) = z.$$ 

It is straightforward that

$$\varphi = \varphi_1^{e_1} \varphi_2^{e_2} \varphi_3^{e_3};$$
and that this expression is unique. This proves that

\[ \text{Inn}G \] is free abelian of rank 3

and hence

\[ \text{Inn}G \neq IA(G) \]

for \( G \) free nilpotent of class 2 and rank 3.

### 2.4 Connections with homotopy theory

We may think of \( IA(G) \) as the group of automorphisms of \( G \) inducing the identity on \( H_1 G \), the first homology group of \( G \). Denote by \( Aut_\ast G \) the group of automorphisms of \( G \) that induce the identity on all homology groups. It is not hard to show that \( \text{Inn}G \leq \text{Aut}_\ast G \leq IA(G) \). It would be worthwhile to attempt the proof that if \( G \) and \( H \) are suitable nilpotent groups in the same localization genus, \( \text{Aut}_\ast G \) and \( \text{Aut}_\ast H \) also belong to the same localization genus.

In fact, the main motivation for theorem 2.3.3 in group theory comes from a related result by Maruyama [11] in homotopy theory involving some version of \( \text{Aut}_\ast G \): let \( X \) be a simply connected CW-complex and denote by \( \varepsilon_0(X) \) the group of homotopy classes of self-homotopy equivalences of \( X \) that induce the identity on all homology groups. E. Dror and A. Zabrodsky proved that \( \varepsilon_0(X) \) is nilpotent [5], so its \( p \)-localization makes sense. Maruyama’s result is that the homomorphism

\[ \varepsilon_0(X) \to \varepsilon_0(X_{(p)}) \]
obtained by localizing each homotopy class is in fact the localization homomorphism of nilpotent groups

$$\varepsilon_0(X) \to \varepsilon_0(X)(p).$$

It is important to observe that our result does not follow from Maruyama’s. If we attempted to derive our theorem from his, we would have to consider a $CW$-complex $X$ whose homotopy type is that of a $K(G, 1)$, with $G$ finitely generated, torsion-free, nilpotent, and metabelian. Already, $X$ is not simply connected so his result does not apply. In addition, his theorem is about the group of homotopy classes of self-homotopy equivalences of $X$ inducing the identity on homology, a group that corresponds to $Aut_\ast G$, which may be smaller than $IA(G)$ in general.
Chapter 3

Examples

3.1 Background

There is a very useful technique developed by Pickel [14] involving certain multilinear forms to study the so-called “one-relator nilpotent groups.” These are finitely generated nilpotent groups of class $c$, each arising as a quotient of a free nilpotent group modulo an infinite cyclic group generated by an element of $\gamma_c$ which is not a proper power. He uses this technique to study such groups, along with their $p$-completions. We will use it to discuss the localization genus of two nilpotent groups of class 4. Pickel’s results in $p$-completions carry over to $p$-localizations mainly because the $p$-localization of a finitely generated, torsion-free nilpotent group $G$, generated by $X$, is a $\mathbb{Z}_{(p)}$-group; meaning that $G_{(p)}$ is generated by $X$ over the $p$-local integers $\mathbb{Z}_{(p)}$.

To explain the details, let $F$ be a free nilpotent group of class $c$ on $n$ generators. Let $a$ be an element of $\gamma_c F$, where $a$ is not a proper power.
Choose a basis \( \{ \omega_{\alpha} \} \) for the free abelian group \( \gamma_c F \) (for example, the \( \omega_{\alpha} \) can be chosen to be the basic commutators of weight \( c \)), and write
\[
a! = \prod_{\alpha} \omega_{\alpha}^{j_{\alpha}}, \quad (j_{\alpha} \in \mathbb{Z}).
\]
Define a map \( \varphi_a : \gamma_c F \to \mathbb{Z} \) on the basis of \( \gamma_c F \) as
\[
\varphi_a(\omega_{\alpha}) = j_{\alpha}.
\]
\( \varphi_a \) clearly extends to a homomorphism.

Now consider the function \( \beta \) from the cartesian product of abelian groups \( F/F' \times \cdots \times F/F' \) (\( c \) copies) to \( \gamma_c F \) given by
\[
\beta(\overline{f_1}, \ldots, \overline{f_c}) = [f_1, \ldots, f_c].
\]
\( \beta \) is a well-defined, multilinear function (see lemma 7.3 in [14]). Associate to \( a \) the multilinear function
\[
f_a = \varphi_a \circ \beta : F/F' \times \cdots \times F/F' \ (c \text{ times}) \to \mathbb{Z}. \tag{3.1}
\]
We show that the function \( f_a \) corresponds to a unique \( a \) in \( \gamma_c F \). To see this, observe that the image of \( \beta : F/F' \times \cdots \times F/F' \to \gamma_c F \) generates \( \gamma_c F \). Next, assume that \( f_a = f_{a'} \) where \( a \) and \( a' \) lie in \( \gamma_c F \). That is,
\[
\varphi_a \circ \beta = \varphi_{a'} \circ \beta.
\]
We will show that \( \varphi_a \) and \( \varphi_{a'} \) agree on the left-normed commutators of length \( c \) (and therefore agree everywhere on \( \gamma_c F \)). Indeed, let \( x = [x_1, \ldots, x_c] \) be a typical left-normed commutator of length \( c \). Choose \( y \in F/F' \times \cdots \times F/F' \) such that \( \beta(y) = x \). Thus
\[
\varphi_a(x) = \varphi_a \circ \beta(y) = \varphi_{a'} \circ \beta(y) = \varphi_{a'}(x).
\]
This proves that $\varphi_a = \varphi_{a'}$. By the definition of the $\varphi$'s, we conclude that $a = a'$.

Following Pickel, we call $f_a$ the $c$-form associated with $a \in \gamma_c F$.

For elements $a$ and $b$ in $\gamma_c F$ which are not proper powers, we say that the associated $c$-forms $f_a$ and $f_b$ are equivalent if there exists and automorphism $\varphi$ of $F/F'$ such that

$$f_a = f_b \circ (\varphi \times \cdots \times \varphi). \quad (3.2)$$

Similarly, each $a \in \gamma_c F(p)$ gives rise to a $c$-form

$$f_a : F(p)/F(p) \times \cdots \times F(p)/F(p) \ (c \text{ times}) \to \mathbb{Z}(p).$$

Given $a$ and $b$ in $\gamma_c F(p)$ we say that $f_a$ is $p$-equivalent to $f_b$ if there is an automorphism $\varphi$ of $F(p)/F'(p)$ such that

$$f_a = f_b \circ (\varphi \times \cdots \times \varphi).$$

Now, any automorphism of $F/F'$ can be represented by an invertible matrix $\varphi_{ij}$ with integral coefficients, relative to the generating set $x_1, \ldots, x_n$ of $F$. The map given by

$$x_i \mapsto \prod_j x_j^{v_{ij}} \quad (3.3)$$

clearly extends to an endomorphism of $F$ and induces the given automorphism of $F/F'$. Lemma 7.2 in [14] gives that the map on the $x_i$ is in fact an automorphism of $F$. The proof of lemma 7.2 depends on the fact that an endomorphism is completely determined by its action on the generators of the group. Since endomorphisms of $F(p)$ and $(F/F')(p)$ are also determined by their actions on the $p$-generators, we have a similar statement for the
localized groups. In the localized case the invertible matrix $\varphi_{ij}$ has entries in $\mathbb{Z}_p$ and the map $3.3$ is an automorphism of $F(p)$.

Consider again $a$ and $b$ in $\gamma_c F$. Proposition 7.1 in [14], which is also valid for localized groups, gives that $F/gp(a)$ is isomorphic to $F/gp(b)$ (resp. $(F/gp(a))_p$ is isomorphic to $(F/gp(b))_p$) if and only if there is an automorphism of $F$ (resp. an automorphism of $F_p$) sending $a$ to $b^\mu$ where $\mu$ is 1 or -1 (resp. $\mu$ is a $p$-local unit).

Suppose that $f_a$ is equivalent (resp. $p$-equivalent) to $f_b$. By definition there is an automorphism $\varphi$ of $F/F'$ (resp. $(F/F')_p$) such that equation 3.2 holds. Let $\hat{\varphi}$ be the “lift” of $\varphi$ given by 3.3, which is an automorphism of $F$. For each $(\overline{g_1}, \ldots, \overline{g_c})$ in the $c$-fold cartesian product of copies of $F/F'$, we have

$$f_a(\overline{g_1}, \ldots, \overline{g_c}) = f_b \left( \overline{\varphi(g_1)}, \ldots, \overline{\varphi(g_c)} \right).$$

By definition (see 3.1) we conclude that

$$\varphi_a [g_1, \ldots, g_c] = \varphi_b \hat{\varphi} [g_1, \ldots, g_c].$$

This means that $\hat{\varphi}$ must send $a$ to $b$ in $\gamma_c$. Notice that the converse is also true: if $\varphi$ is an automorphism of $F$ (resp. $F_p$) such that $\hat{\varphi}$ sends $a$ to $b$, then $f_a$ and $f_b$ are equivalent (resp. $p$-equivalent). Finally, it is clear that the form associated to the element $a^\mu$ where $\mu = 1, -1$ (resp. $\mu$ is a $p$-local unit) is $\mu f_a$. All this gives the following lemma (analogous to proposition 8.1 in [14]).

**Lemma 3.1.1.** $F/gp(a)$ is isomorphic (resp. $p$-isomorphic) to $F/gp(b)$ if and only if $f_a$ is equivalent (resp. $p$-equivalent) to $\mu f_b$ where $\mu$ is 1 or -1 (resp. a $p$-local unit).
We will be working with one-relator nilpotent groups of class 4 that are also metabelian. The advantage of the 4-forms associated to these groups is that they can be reduced to symmetric bilinear forms. The following result (see lemma 9.1 in [14]) is very useful in this direction. We give a detailed proof here.

**Lemma 3.1.2.** Let $F$ be a free nilpotent group of class $c$, which is also metabelian. Then any left-normed commutator of length $c$ satisfies

$$[a_1, a_2, a_3, \ldots, a_c] = [a_1, a_2, a_{\sigma(3)}, \ldots, a_{\sigma(c)}]$$

where $\sigma$ is a permutation of $\{3, 4, \ldots, c\}$. Put another way, left-normed commutators of length $c$ are symmetric in the last $c - 2$ entries.

**Proof.** We begin by defining an action of the group ring $\mathbb{Z}(F/F')$ on $F'$. For $\overline{g} \in F/F'$ and $x \in F'$ define

$$\overline{g}x = x^g \in F'.$$

Similarly, integers act on $F'$ by exponentiation. It suffices to check that this alleged action restricted to the basis $F/F'$ of the group ring is well defined. To show that this definition does not depend on the chosen representative $g$ suppose that

$$\overline{g} = \overline{g'}$$

in $F/F'$. Then

$$x^g = x^{g'} \iff g^{-1}xg = g'^{-1}xg' \iff g'^{-1}xgg'^{-1} = x \iff (gg'^{-1})^{-1}xgg'^{-1} = x.$$ 

But this clearly holds since both $gg'^{-1}$ and $x$ lie in the abelian group $F'$. Notice that:
\[
[a_1, a_2]^{(-1+a_3)} = [a_1, a_2]^{-1} [a_1, a_2]^{a_3} = [a_1, a_2]^{-1} [a_1, a_2] [a_1, a_2] = [a_1, a_2, a_3].
\]

\[
[a_1, a_2, a_3]^{(-1+a_4)} = [a_1, a_2, a_3, a_4].
\]

\[
[a_1, a_2, a_3, \ldots, a_k]^{(-1+a_{k+1})} = [a_1, a_2, a_3, \ldots, a_{k+1}].
\]

Hence, the left-normed commutator

\[
[a_1, a_2, \ldots, a_c] = [[[\ldots [a_1, a_2], a_3], \ldots, a_c]
\]

can be rewritten as

\[
\left( \ldots \left( \left( [a_1, a_2]^{(-1+a_3)} \right)^{(-1+a_4)} \right)^{(-1+a_5)} \ldots \right)^{(-1+a_c)} = [a_1, a_2]^{(-1+a_3) \cdots (-1+a_c)}.
\]

Finally, let \( \sigma \) be a permutation of \( \{3, \ldots, c\} \). Since the group ring \( \mathbb{Z}(F/F') \) is commutative we conclude that

\[
(-1+a_3) \cdots (-1+a_c) = (-1+a_{\sigma(3)}) \cdots (-1+a_{\sigma(c)}),
\]

so the lemma follows.

Let \( F = gp(x, y) \) be the two-generator free nilpotent group of class 4 and consider the basic sequence of basic commutators on \( \{x, y\} \),

\[
\{ x, y, c_{21}, c_{212}, c_{211}, c_{2122}, c_{2112}, c_{2111} \}
\]

where \( c_{21} = [y, x] \), \( c_{212} = [y, x, y] \), \( c_{211} = [y, x, x] \), \( c_{2122} = [y, x, y, y] \), \( c_{2112} = [y, x, x, y] \), and \( c_{2111} = [y, x, x, x] \). (We will use this notation repeatedly).

We now describe how to associate a symmetric bilinear form to a quotient of \( F \) by a cyclic subgroup of \( \gamma_4 F \).

The following is a simple (but crucial) observation:
Lemma 3.1.3. \( F \) is metabelian.

Proof. This follows from the fact that \( F \) has a very low rank. Since \( c_{21} \) is the only basic commutator of weight 2, all basic commutators commute. Therefore, \( F' \) is abelian and \( F \) is metabelian.

A basis for \( \gamma_4 F \) is given by

\[
\{c_{2122}, c_{2112}, c_{2111}\}.
\]

Choose

\[
a! = c_{2122}^{a_2} c_{2112}^{a_2} c_{2111}^{a_3}
\]

in \( \gamma_4 F \). Then the function

\[
f_a : F/F' \times F/F' \times F/F' \times F/F' \to \mathbb{Z}
\]

is a 4-form. By lemma 3.1.2, this 4-form induces the symmetric bilinear form

\[
\hat{f}_a : F/F' \times F/F' \to \mathbb{Z}
\]

\[
\hat{f}_a(y_1, y_2) = f_a(x, y_1, y_2).
\]

Let

\[
F_a = F/gp(a).
\]

\( \hat{f}_a \) is the symmetric bilinear form associated to \( F_a \).

Consider two elements \( a \) and \( b \) of \( \gamma_4 F \) and their corresponding symmetric bilinear forms \( \hat{f}_a \) and \( \hat{f}_b \). Let \( A_a \) and \( A_b \) be the symmetric integral matrices associated with \( \hat{f}_a \) and \( \hat{f}_b \) respectively, relative to the basis of \( F/F' \) induced by the generators of \( F \). The notion of equivalence (\( p \)-equivalence) of symmetric bilinear forms now translates into equivalence (\( p \)-equivalence)
of their corresponding matrices. We may therefore say that \( \hat{f}_a \) is equivalent \((p\text{-equivalent})\) to \( \hat{f}_b \) if there is a \( \mathbb{Z} \)-invertible \((\mathbb{Z}(p)\text{-invertible})\) matrix \( M \) such that

\[
A_a = M A_b M^t.
\]

The matrices \( A_a \) and \( A_b \) themselves may be regarded as equivalent \((p\text{-equivalent})\).

The following result is analogous to lemma 3.1.1, replacing forms with matrices.

**Lemma 3.1.4.** \( F_a \) is isomorphic \((p\text{-isomorphic})\) to \( F_b \) if and only if \( \mu A_a \) is equivalent \((p\text{-equivalent})\) to \( A_b \) where \( \mu \) is 1 or -1 (a unit in \( \mathbb{Z}(p) \)).

According to the definition, to show that two integral symmetric matrices are equivalent over \( \mathbb{Z}(p) \) one needs to find an invertible matrix \( M \) with entries in \( \mathbb{Z}(p) \). However, the following result of Bokor reveals that our search can be restricted to integral matrices. We include the proof given in [3].

Following Bokor, call two integral symmetric matrices *weakly equivalent* \((\text{weakly } p\text{-equivalent})\) if they are equivalent up to a unit in \( \mathbb{Z} \) (in \( \mathbb{Z}(p) \)).

**Lemma 3.1.5.** (lemma 1 in [3]) Two integral symmetric matrices \( G \) and \( H \) are weakly \( p\text{-equivalent} \) if and only if there exists an integral matrix \( A \) (not necessarily invertible over \( \mathbb{Z} \) but with non-zero determinant) and an integer \( m \) such that

\begin{itemize}
  \item \( m \) and \( p \) are relatively prime,
  \item \( \det A \) and \( p \) are relatively prime,
  \item \( mG = AHA^t \).
\end{itemize}
Proof. Assume that $G$ and $H$ are weakly $p$-equivalent. Then there exists a $\mathbb{Z}_p$-invertible matrix $A'$ and a unit $m'$ of $\mathbb{Z}_p$ such that

$$m'G = A'H(A')^t.$$  

We now construct $m$ and $A$ with the desired properties.

Let $k$ be the least common multiple of the denominators of the entries of $A'$ and the denominator of $m'$. Observe that the prime decomposition of $k$ does not include $p$. Therefore, $k$ and $p$ are relatively prime and $k$ is a unit in $\mathbb{Z}_p$. Also $k^2$ and $p$ are relatively prime. Let

$$m = k^2m'$$

and

$$A = kA'.$$

$m$ is an integer which is relatively prime to $p$. $A$ is an integral matrix, invertible over $\mathbb{Z}_p$. Therefore the integer $\det A$ is a unit in $\mathbb{Z}_p$, so that $\det A$ is relatively prime to $p$. Finally,

$$mG = k^2m'G = kA'HK(A')^t = AHA^t.$$  

Conversely, assume now that

$$mG = AHA^t$$

where $\det A$ and $m$ are each relatively prime to $p$. These two integers are units in $\mathbb{Z}_p$ so that $G$ and $H$ are weakly $p$-equivalent. 

\[\square\]
3.2 Remeslennikov’s groups

We now describe two non-isomorphic, class 4 nilpotent and metabelian groups which lie in the same localization genus.

In the category of class 4 nilpotent groups let

\[ F = \langle x, y \rangle, \]

\[ F_S = \langle x, y; c_{2122}^3 c_{2112}^2 c_{2111}^2 \rangle, \]

and

\[ F_T = \langle x, y; c_{2122}^6 c_{2112} c_{2111} \rangle. \]

**Remark.** Neither \( F_S \) nor \( F_T \) is isomorphic to \( F \). On the one hand, \( \gamma_4 F \) is free abelian of rank 3, freely generated by \( \{ c_{2122}, c_{2112}, c_{2111} \} \). On the other, both \( \gamma_4 F_S \) and \( \gamma_4 F_T \) are free abelian of rank 2: it is easy to show that \( \gamma_4 F_S \) is freely generated by \( \{ c_{2122}^3 c_{2112} c_{2111}, c_{2122} \} \) and \( F_T \) is freely generated by \( \{ c_{2122}^6 c_{2112}, c_{2122} \} \).

Consider the symmetric bilinear forms

\[ \hat{f}_S, \hat{f}_T : F/F' \times F/F' \to \mathbb{Z} \]

associated to \( F_S \) and \( F_T \) respectively.

Let \( M_S = (m_{ij}) \) and \( M_T = (m'_{ij}) \) be the symmetric matrices of these forms with respect to the ordered basis \( \{ \bar{x}, \bar{y} \} \) of the \( \mathbb{Z} \)-module \( F/F' \). \( M_S \) is found as follows:

- \( m_{11} = \hat{f}_S(\bar{x}, \bar{x}) = f_S(\bar{y}, \bar{x}, \bar{x}, \bar{x}) = 2 \)
- \( m_{12} = \hat{f}_S(\bar{x}, \bar{y}) = f_S(\bar{y}, \bar{x}, \bar{x}, \bar{y}) = 1 \)

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• $m_{21} = \hat{f}_s(\bar{y}, \bar{x}) = f_s(\bar{y}, \bar{x}, \bar{y}) = f_s(\bar{y}, \bar{x}, \bar{y}) = 1$ (see lemma 3.1.2)

• $m_{22} = \hat{f}_s(\bar{y}, \bar{y}) = f_s(\bar{y}, \bar{x}, \bar{y}) = 3$

Similarly, we find $M_T$:

• $m'_{11} = \hat{f}_t(\bar{x}, \bar{x}) = f_t(\bar{y}, \bar{x}, \bar{x}) = 1$

• $m'_{12} = \hat{f}_t(\bar{x}, \bar{y}) = f_t(\bar{y}, \bar{x}, \bar{x}, \bar{y}) = 1$

• $m'_{21} = \hat{f}_t(\bar{y}, \bar{x}) = f_t(\bar{y}, \bar{x}, \bar{y}, \bar{x}) = f_t(\bar{y}, \bar{x}, \bar{y}) = 1$

• $m'_{22} = \hat{f}_t(\bar{y}, \bar{y}) = f_t(\bar{y}, \bar{x}, \bar{y}) = 6$

In an attempt to diagonalize each matrix over the integers, we perform row/column elementary operations:

1. $M_S|I = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \\ 2 & -2 & 1 & 0 \\ -2 & 12 & 0 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -2 & 1 & 0 \\ 1 & -6 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ -2 & 10 & 0 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 10 & 1 & -2 \end{pmatrix} = D|Q^t$ where $Q^t M_S Q = D$.

   Since $\det Q = -2$, a unit in $\mathbb{Z}_{(p)}$ for $p \neq 2$, this diagonalization process is valid for $p \neq 2$. It is not valid over $\mathbb{Z}(2)$ or over the integers.

2. $M_T|I = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 6 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 5 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 5 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 5 & -1 & 1 \end{pmatrix} = D|Q^t$ where $Q^t M_T Q = D$. In this case $\det Q = 1$, so this diagonalization process is valid over $\mathbb{Z}$, as well as over $\mathbb{Z}_{(p)}$ for every $p$.  

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Lemma 3.2.1. The matrices \[
\begin{pmatrix}
1 & 1 \\
1 & 6
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
2 & 1 \\
1 & 3
\end{pmatrix}
\] are not weakly equivalent over \(\mathbb{Z}\).

Proof. By the above diagonalization process it suffices to show that \[
\begin{pmatrix}
2 & 1 \\
1 & 3
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 \\
0 & 5
\end{pmatrix}
\] are not weakly equivalent over the integers. Suppose first that there is a matrix \(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) with \(\det M = 1\) and such that
\[
\begin{pmatrix}
2 & 1 \\
1 & 3
\end{pmatrix} = M^t \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} M.
\]
This gives
\[
a^2 + 5b^2 = 3,
\]
an equation with no integer solutions.

To show that \(-\begin{pmatrix} 1 & 1 \\ 1 & 6 \end{pmatrix}\) and \(\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\) are not equivalent either, we diagonalize the first matrix to get \(-\begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}\) and assume that
\[
\begin{pmatrix}
2 & 1 \\
1 & 3
\end{pmatrix} = M^t \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix} M.
\]
This yields
\[-b^2 - 5d^2 = 3.
\]
Again, an impossible situation. This completes the proof. \qed
Lemma 3.2.2. The matrices \( \begin{pmatrix} 1 & 1 \\ 1 & 6 \end{pmatrix} \) and \( \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \) are weakly \( p \)-equivalent for all \( p \).

Proof. Assume first that \( p \neq 2 \). As we did in the computations above, we find that \( \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 1 \\ 1 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \), and these are valid (strong) \( p \)-equivalences for all \( p \neq 2 \). Since
\[
\begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}
\]
and 2 is a unit in \( \mathbb{Z}_{(p)} \), the lemma follows in this case.

It suffices to prove now that \( \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \) are weakly 2-equivalent. By lemma 3.1.5 the search for an invertible “transition” matrix \( Q \) and a unit \( \mu \) of \( \mathbb{Z}_{(2)} \) can be restricted to the integers. Let \( Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) be an integral matrix with odd determinant and \( \mu \) and odd integer. Consider the integral matrix equation
\[
\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.
\]
This gives:
\[
a^2 + 5c^2 = 2\mu \\
ab + 5cd = \mu \\
b^2 + 5d^2 = 3\mu.
\]
The first of these equations has no solutions for \( \mu = 1 \) so we try the next odd number \( \mu = 3 \) (notice that \( \mu \) can never be negative). For this value of \( \mu \) we
get

\[ a^2 + 5c^2 = 6 \]
\[ ab + 5cd = 3 \]
\[ b^2 + 5d^2 = 9. \]

The first equation is satisfied by \( c = 1 = a \) and the third by \( d = 0, b = 3 \). These values also satisfy the second equation. Put

\[
M = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}.
\]

Since \( \det M = -3 \), \( M \) is invertible over \( \mathbb{Z}_{(2)} \) and

\[
M^t \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} M = 3 \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.
\]

This completes the proof.

\textbf{Corollary 3.2.3.} \textit{Remeslennikov’s groups} \( F_S \) \textit{and} \( F_T \) \textit{are not isomorphic but they lie in the same localization genus.}

In fact, if two integral symmetric matrices are non-singular (by this we mean that they have non-zero determinant, although they need not be invertible over \( \mathbb{Z} \)), they are equivalent over the \( p \)-local integers if and only if they are equivalent over the \( p \)-adic integers (See [3] and reference [2] there). Hence, we have also shown that the groups \( F_S \) and \( F_T \) lie in the same completion genus.
3.3  IA-automorphisms of Remeslennikov’s groups

Consider again the class 4 nilpotent groups

\[ F = \langle x, y \rangle , \]

\[ F_S = \langle x, y; c_{2122} c_{2112}^3 c_{2111}^2 \rangle , \]

and

\[ F_T = \langle x, y; c_{2122} c_{2112} c_{2111}^6 \rangle . \]

We record some facts about their groups of IA-automorphisms. We need the following lemma:

**Lemma 3.3.1.** \( F_S \) and \( F_T \) are torsion-free.

**Proof.** We first show that that \( F'_S \) and \( F_S/F'_S \) are torsion-free. Let

\[ x = c_{21}^a c_{211}^b c_{212}^d c_{2111}^e c_{2122}^f \in F'_S \]

and suppose that there exists a positive integer \( n \) with

\[ x^n = c_{21}^{na} c_{211}^{nb} c_{212}^{nd} c_{2111}^{ne} c_{2122}^{nf} = 1 \in F'_S . \]

Then there exists an integer \( \omega \) such that

\[ c_{21}^{na} c_{211}^{nb} c_{212}^{nd} c_{2111}^{ne} c_{2122}^{nf} = c_{2122}^{3\omega} c_{2112}^{2\omega} c_{2111}^\omega \in F . \]

From this we see that \( a = b = c = 0, d = 3e, \) and \( f = 2e . \) Hence

\[ x = c_{2122}^{3e} c_{2112}^{2e} c_{2111} = 1 \in F_S . \]

This proves that \( F'_S \) is torsion-free.
Next observe that $F_S/F'_S$ is free abelian of rank 2, so it is torsion-free. Finally let $x \in F_S$ such that $x^n = 1 \in F_S$ for some positive $n$. Then $\overline{x}^n = 1 \in F_S/F'_S$. As $F_S/F'_S$ is torsion-free, $\overline{x} = 1 \in F_S/F'_S$ so that $x \in F'_S$. Since $F'_S$ is torsion-free, this implies that $x = 1$, completing the argument that $F_S$ is torsion-free.

For $F_T$ it suffices to show that $F'_T$ is torsion-free; the rest of the argument is the same. To see this let

$$x = c_1^a c_2^b c_3^c c_4^d c_5^e c_6^f \in F'_T,$$

and assume that $x^n = 1$ with $n \neq 0$. Then, there exists an integer $\omega$ such that

$$c_1^a c_2^b c_3^c c_4^d c_5^e c_6^f = c_2^6 c_3^\omega c_4^\omega c_5^\omega c_6^\omega \in F.$$

By equating exponents we get $a = b = c = 0, d = 6e$, and $f = e$. Hence

$$x = c_2^6 c_3^e c_5^e c_6^e \in F'_T.$$

This completes the proof.

Let $G$ denote $F_S$ or $F_T$. By corollary 1.2.11, $IA(G)$ is torsion-free and nilpotent of class 3. Since nilpotent groups of class 3 are always metabelian, so is $IA(G)$. As a corollary of theorem 2.3.3 we have:

**Lemma 3.3.2.** $IA(F_S)$ and $IA(F_T)$ are finitely generated, torsion-free, metabelian, and nilpotent of class 3, which lie in the same localization genus.

Using computational techniques based on our $\delta$-sequences from chapter 2, it is possible (but tedious) to find normal forms for $IA(F_S)$ and $IA(F_T)$ involving, in each case, generators of the form

$$\varphi_i(x) = xc_i, \quad \varphi_i(y) = y$$

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and

$$\varphi_j(x) = x, \quad \varphi_j(y) = yc_j;$$

where $c_i$ and $c_j$ range over the free generators of the free abelian groups $F'_S$ and $F'_T$, respectively. These normal forms lead to presentations for $IA(F_S)$ and $IA(F_T)$. Even though it is still unclear from this whether $IA(F_S)$ and $IA(F_T)$ are isomorphic or not, our results certainly imply that $IA(F_S)$ and $IA(F_T)$ lie in the same localization genus.
Bibliography


