IA-automorphisms of groups with almost constant upper central series

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Abstract. Let $G$ be any group for which there is a least $j$ such that $Z_j = Z_{j+1}$ in the upper central series. Define the group of $j$-central automorphisms as the kernel of the natural homomorphism from $\text{Aut}(G)$ to $\text{Aut}(G/Z_j)$. We offer sufficient conditions for $IA(G)$ to have a useful direct product structure, and apply our results to certain finitely generated center-by-metabelian groups.

1. Introduction

The $IA$-group of a group $G$, denoted by $IA(G)$, consists of those automorphisms that induce the identity on $G/G'$, where $G'$ is the commutator subgroup of $G$. The investigation of the $IA$-group has been of interest in different contexts. P. Hall, for example, has shown that the $IA$-group of a nilpotent group of class $c$ is nilpotent of class $c - 1$ (see [5]); and M. Zyman has remarked that if $G$ is finitely generated nilpotent, so too is $IA(G)$ (see [8]).

A result of particular interest here is due to S. Bachmuth (see [1]), who has shown that the $IA$-group of a free metabelian group of rank two is equal to its inner automorphism group. He has also shown that this is not the case when the rank is larger than two. In this paper we study the $IA$-group of a group $G$ for which the upper central series stalls at some point. This means that there exists a least positive integer $j$ for which $Z_j = Z_{j+1}$. We refer to these groups as $H_j$-groups, whose examination was inspired by an example of P. Hall in [6]. We discuss this example in §4.

We define the group of $j$-central automorphisms of $G$, denoted by $\text{Aut}_{c_j}(G)$, as the kernel of the natural homomorphism from $\text{Aut}(G)$ to $\text{Aut}(G/Z_j)$, where $Z_j$ is a term of the upper central series. In Lemma 3.1 we prove that for $G$ an $H_j$-group, the subgroup of $\text{Aut}(G)$ generated by the inner automorphisms together with the $j$-central automorphisms is a direct product of these two groups, modulo $\text{Aut}_{c_j - 1}(G)$. Let $\text{Aut}_{c_j}(G) = IA(G) \cap \text{Aut}_{c_j}(G)$. In Theorem 3.2, we give sufficient conditions

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for the group of $IA$-automorphisms of an $\mathcal{H}_j$-group to equal the direct product of the inner automorphisms with $Aut_{x_j}(G)$, modulo $Aut_{x_{j-1}}(G)$. The main result is:

**Theorem 3.3.** Let $G$ be a group for which there is a least $j$ such that $Z_j = Z_{j+1}$ in the upper central series. If $IA(G/Z_j) = Inn(G/Z_j)$, then

$$IA(G)/Aut_{x_{j-1}}(G) = (Inn(G)Aut_{x_{j-1}}(G)/Aut_{x_{j-1}}(G)) \times (Aut_{x_j}(G)/Aut_{x_{j-1}}(G)).$$

When $Z_1 = Z_2$, we get the very pleasing result:

**Corollary 3.4** Let $G$ be a group such that $Z_1 = Z_2$ in the upper central series. If $IA(G/\zeta(G)) = Inn(G/\zeta(G))$, then

$$IA(G) = Inn(G) \times Aut_{x_1}(G).$$

The paper is organized as follows: In §2 we provide definitions, notation, and a preliminary lemma; while §3 is devoted to the proof of our theorem. In §4 we consider some examples, motivated by the existence of a family of finitely generated metabelian groups whose $IA$-group is not finitely generated [2]. As further evidence of the complexity of solvable groups, we end §4 with the construction of continuously many finitely generated center-by-metabelian groups whose group of $IA$-automorphisms is not finitely generated.

2. Preliminary discussion

We write $G = gp(A)$ to express the fact that $A$ is a generating set for $G$. We write a typical commutator as $[g, h] = g^{-1}h^{-1}gh$.

For any group $G$ define the group of $IA$-automorphisms (or $IA$-group) to be the kernel of the natural homomorphism from $Aut(G)$ to $Aut(G/G')$. Thus,

$$IA(G) = \{\alpha \in Aut(G) : g^{-1}(g\alpha) \in G' \ (g \in G)\}.$$ 

For any group $G$, $Inn(G) \leq IA(G)$ and $G/\zeta(G) \cong Inn(G)$. Throughout this paper, the inner automorphism induced by $x \in G$ will be denoted as $\alpha_x$.

For any positive integer $j$, we define the group of $j$-central automorphisms of $G$, denoted by $Aut_{c_j}(G)$, to be the kernel of the natural homomorphism from $Aut(G)$ to $Aut(G/Z_j)$. A $j$-central automorphism acts as the identity on $G$ modulo $Z_j$. Thus:

$$Aut_{c_j}(G) = \{\alpha \in Aut(G) : g^{-1}(g\alpha) \in Z_j \ (g \in G)\}.$$ 

We use the notation $Aut_c(G) = Aut_{c_1}(G)$ and we refer to $Aut_c(G)$ as the group of central automorphisms of $G$. An alternate definition of $Aut_{c_j}(G)$ is given by the following lemma:

**Lemma 2.1.** For any group $G$,

$$Aut_{c_j}(G) = \{\varphi \in Aut(G) : [\varphi, \alpha_x] \in Aut_{c_{j-1}}(G) \text{ for all } \alpha_x \in Inn(G)\}.$$
Proof.
\[ \varphi \in Aut_{c_j}(G) \iff g^{-1}(g\varphi) \in Z_j \text{ for all } g \in G \]
\[ \iff x^{-1}(x\alpha_{g^{-1}(g\varphi)}) = x^{-1}(x\alpha_g^{-1}\alpha_g^\varphi) \in Z_j \text{ for all } x, g \in G \]
\[ \iff \alpha_g^{-1}\alpha_g^\varphi = [\alpha_g, \varphi] \in Aut_{c_j-1}(G) \text{ for all } g \in G. \]

As stated in §1, a group \( G \) is an \( H_j \)-group if there exists a least positive integer \( j \) for which \( Z_j = Z_{j+1} \).

Since there is a natural correspondence between \( Aut_{z_j}(G) = Aut_{c_j}(G) \cap IA(G) \) and \( G' \cap Z_j \), we can also characterize \( Aut_{z_j}(G) \) as
\[ Aut_{z_j}(G) = \{ \alpha \in AutG : g^{-1}(g\alpha) \in Z_j \cap G' (g \in G) \}. \]

We remark that every nilpotent group of class \( c \) is trivially an \( H_c \)-group, since \( G = Z_c = Z_{c+1} = \cdots \).

A group \( G \) is termed center-by-metabelian if \( G/\zeta(G) \) is metabelian. \( G \) is center-by-metabelian if and only if \( G'' \leq \zeta(G) \), where \( G'' = [G', G'] \). The variety of center-by-metabelian groups contains interesting examples of \( H_1 \)-groups, some of which will be discussed in §4.

3. Proof of the main result

Lemma 3.1. Let \( G \) be an \( H_j \)-group. Then, modulo \( Aut_{c_j-1}(G) \),
\[ gp(\text{Inn}(G), Aut_{c_j}(G)) \cong (\text{Inn}(G)Aut_{c_j-1}(G)) \times Aut_{c_j}(G). \]

Proof. By Lemma 2.1,
\[ [\text{Inn}(G), Aut_{c_j}(G)] \leq Aut_{c_j-1}(G) \]
for any group \( G \).

To prove that \( \text{Inn}(G) \cap Aut_{c_j}(G) \leq Aut_{c_j-1}(G) \), suppose that \( \alpha_x \in \text{Inn}(G) \cap Aut_{c_j}(G) \), where \( g\alpha_x = g[g, x] \). Then for all \( g \in G \), \([g, x] \in Z_j \). This implies \( x \in Z_{j+1} \). Since \( Z_j = Z_{j+1} \), we have \( x \in Z_j \), and thus \([g, x] \in Z_j \) for all \( g \in G \). Hence, \( \text{Inn}(G)Aut_{c_j-1}(G) \cap Aut_{c_j}(G) = 1 \) modulo \( Aut_{c_j-1}(G) \), and our result follows.

Note that if \( G \) is not an \( H_j \)-group, then it is not necessarily the case that \( \text{Inn}(G) \cap Aut_{c_j}(G) \leq Aut_{c_j-1}(G) \). For example, let
\[ G = \langle a, t_0, t_1, \ldots ; t_0 = 1, [a, t_i] = t_{i-1} (i = 1, 2, \ldots) , [t_i, t_j] = 1 \rangle. \]

Observe that \( Z_j = gp(t_1, \ldots, t_j) \) and \( G \) has a strictly ascending upper central series. In particular \( G \) is not an \( H_j \)-group for any \( j \). Let \( T = gp(t_1, t_2, \ldots) \). Any element of \( G \) can be written uniquely as \( g = a^mt \), for some \( m \in \mathbb{Z} \) and some \( t \in T \). Then, \( g\alpha_{t_{j+1}} = g[g, t_{j+1}] = g[a^m, t_{j+1}] \). Since \([a, t_{j+1}] = t_j \in Z_j \), \([a^m, t_{j+1}] \in Z_j \). Hence \( \alpha_{t_{j+1}} \in Aut_{c_j}(G) \). Moreover, since \( a\alpha_{t_{j+1}} = at_j \), then \( \alpha_{t_{j+1}} \notin Aut_{c_j-1}(G) \).

Our main result is:

Theorem 3.2. Let \( G \) be a group for which there is a least \( j \) such that \( Z_j = Z_{j+1} \) in the upper central series. If \( IA(G/Z_j) = \text{Inn}(G/Z_j) \), then, modulo \( Aut_{z_j-1}(G) \),
\[ IA(G) = (\text{Inn}(G)Aut_{z_j-1}(G)) \times Aut_{z_j}(G). \]
Proof. It suffices to show that $\text{IA}(G) = \text{Inn}(G) \text{Aut}_{z_j}(G)$. Then, by Lemma 2.1 and the proof of Lemma 3.1, $\text{IA}(G) = \text{Inn}(G) \times \text{Aut}_{z_j}(G)$, modulo $\text{Aut}_{z_j-1}(G)$.

Let $\varphi \in \text{IA}(G)$. Then $x^{-1}(x\varphi) \in G' \leq G'Z_j$ for all $x \in G$. Thus,

$$x^{-1}(x\varphi)G'Z_j = G'Z_j \text{ for all } x \in G.$$  

By hypothesis $\text{IA}(G/Z_j) = \text{Inn}(G/Z_j)$, so there exists $g \in G$ such that for all $x \in G$,

$$(x\varphi)Z_j = x^gZ_j.$$  

Hence,

$$x^{-9}(x\varphi) = (x^{-1}(x\varphi)^{-1})^9 \in Z_j,$$

and consequently, $x^{-1}(x\varphi^{-1}) \in Z_j$. I.e.,

$$\varphi^{-1} \in \text{Aut}_{c_j}(G) \cap \text{IA}(G) = \text{Aut}_{z_j}(G).$$

Then $\varphi = \varphi^{-1} \alpha_g \in \text{Aut}_{z_j}(G)\text{Inn}(G)$.

Corollary 3.3. Let $G$ be a group such that $Z_1 = Z_2$ in the upper central series. If $\text{IA}(G/\zeta(G)) = \text{Inn}(G/\zeta(G))$, then $\text{IA}(G) = \text{Inn}(G) \times \text{Aut}_{z_2}(G)$.

4. Examples

For the remainder of the paper, we will be considering $\mathcal{H}_1$-groups; for simplicity, we will refer to them as $\mathcal{H}$-groups, and we will denote $\text{Aut}_c(G)$ and $\text{Aut}_{z_j}(G)$ as $\text{Aut}_c(G)$ and $\text{Aut}_{z_j}(G)$, respectively. Accordingly, The 1-central automorphisms will be simply termed central automorphisms.

Although the group of $\text{IA}$-automorphisms of a two-generator metabelian group is metabelian [4], it need not be finitely generated. There is a family of examples of finitely generated metabelian groups whose group of $\text{IA}$-automorphisms is not finitely generated [2].

As further evidence of the complexity of solvable groups, we provide continuously many finitely generated center-by-metabelian groups whose group of $\text{IA}$-automorphisms is not finitely generated.

The following results play a central role in the discussion of our examples.

Lemma 4.1. Let $G = gp(x_1, \ldots, x_n)$ be a finitely generated $\mathcal{H}$-group such that $\zeta(G) \leq G'$. Suppose that there exists a presentation for $G$ in which every relator is a product of commutators. Assume also that $\zeta(G) = gp(z_1, z_2, \ldots)$. Then each map of the form

$$\varphi_{ij} : x_i \mapsto x_iz_j$$

$$x_k \mapsto x_k \quad (k \neq i)$$

lifts to a central automorphism of $G$, and the set of all $\varphi_{ij}$ generates $\text{Aut}_c(G)$. Furthermore, if $\zeta(G)$ is freely generated by $\{z_1, z_2, \ldots\}$, then $\text{Aut}_c(G)$ is freely generated by the $\varphi_{ij}$.

Proof. Let $G = gp(x_1, \ldots, x_n)$ be a finitely generated $\mathcal{H}$-group, and let $\{z_1, z_2, \ldots\}$ be a generating set for $\zeta(G)$. Since $G$ comes with a presentation in which every relator is a product of commutators, each $\varphi_{ij}$ extends to a map on $G$ that sends defining relators to 1. Hence, each $\varphi_{ij}$ lifts to a homomorphism. Since $G' \geq \zeta(G)$, the inverse for $\varphi_{ij}$ can be constructed; therefore, $\varphi_{ij}$ is a central automorphism.
Next we show that \( \{ \phi_{ij} \} \) generates \( \text{Aut}_c(G) \). Let \( \varphi \in \text{Aut}_c(G) \) and write
\[
x_i \varphi = x_i \prod_{j=1}^{m} z_j^{\alpha_{ij}}.
\]

Since central automorphisms act trivially on \( G' \) and \( G' \geq \zeta(G) \),
\[
x_i \prod_{k,j=1}^{n,m} \varphi_{k,j}^{\alpha_{kj}} = x_i \prod_{j=1}^{m} \varphi_{ij}^{\alpha_{ij}} = x_i \prod_{j=1}^{m} z_j^{\alpha_{ij}} = x_i \varphi
\]
for each \( x_i \). It is easy to show that if the \( \varphi_{ij} \) do not freely generate \( \text{Aut}_c(G) \), then the \( z_i \) do not freely generate \( \zeta(G) \). This completes the proof. \( \square \)

**Lemma 4.2.** Let \( G = gp(x_1, \ldots, x_n) \) be a finitely generated \( \mathcal{H} \)-group such that \( \zeta(G) \leq G' \). Suppose that there exists a presentation for \( G \) in which every relator is a product of commutators. If \( \text{Aut}_c(G) \) is finitely generated, then \( \zeta(G) \) is finitely generated.

**Proof.** Suppose \( \{ \alpha_1, \ldots, \alpha_m \} \) generates \( \text{Aut}_c(G) \). Then
\[
x_i \alpha_j = x_i z_{ij}
\]
with \( i = 1, \ldots, n; j = 1, \ldots, m; \) and \( z_{ij} \in \zeta(G) \).

Suppose now that there exists a \( z \in \zeta(G) \) such that \( z \notin gp(z_{11}, \ldots, z_{nm}) \). Then we can define a central automorphism
\[
\alpha_z : x_1 \mapsto x_1 z
\]
\[
x_i \mapsto x_i \ (i \neq 1)
\]
which is not generated by \( \{ \alpha_1, \ldots, \alpha_m \} \), a contradiction. \( \square \)

**Example 1.** Let
\[
G = F/[F'', F]
\]
b be the free center-by-metabelian group of rank 2, where \( F = gp(x, y) \) is the free group on two generators. Then \( \zeta(G) = G'' \) and \( G/\zeta(G) \) is free metabelian of rank 2. It is well known that a free metabelian group has trivial center. Hence, \( G \) is an \( \mathcal{H} \)-group, with \( \text{Aut}_c(G) = \text{Aut}_z(G) \).

By a theorem of Bachmuth (see [1]), \( IA(G/\zeta(G)) = \text{Inn}(G/\zeta(G)) \), so we may apply Theorem 3.2 to conclude that \( IA(G) = \text{Inn}(G) \times \text{Aut}_c(G) \). Ridley has shown in [7] that \( \zeta(G) \) is free abelian of countably infinite rank. Thus, it follows from Lemmas 4.1 and 4.2 that \( \text{Aut}_c(G) \) is itself free abelian of infinite rank. In particular, we have exhibited a finitely generated center-by-metabelian group whose group of \( IA \)-automorphisms is not finitely generated.

Before proceeding, we state an important lemma. We are unaware of an existing proof in the literature.

**Lemma 4.3.** If \( W = \langle a \rangle \wr \langle t \rangle \) is the wreath product of an infinite cyclic group by another, then \( IA(W) = \text{Inn}(W) \).

**Proof.** There is a standard expanded presentation for \( W \) that is convenient for us to use:
\[
W = \langle a, t, a_i \ (i \in \mathbb{Z}); a_0 = a, a_i^t = a_{i+1}, [a_i, a_j] = 1 \ (i, j \in \mathbb{Z}) \rangle.
\]
By examining this presentation we deduce that $W'$ is free abelian, freely generated by
\[
\{ [t, a]^t \mid (i \in \mathbb{Z}) \}.
\]
It follows that $W'$ can also be regarded as a free module over $\mathbb{Z}[t, t^{-1}]$, generated by the single element $\mu = [t, a]$. The module structure allows us to show that every $IA$-automorphism is an inner automorphism.

Let $\varphi \in IA(G)$. Then
\[
\varphi : a \mapsto a\mu^{s_a},
\]
where $r_a, r_t \in \mathbb{Z}[t, t^{-1}]$.

Applying the commutator calculus and using the module structure of $W'$ we have:
\[\begin{align*}
(\text{i}) \quad [t, \mu] &= [t, [t, a]] = [t, a^{-1}a] = [t, a][t, a^{-1}] = \mu^{-t} = \mu^{-t}, \text{ and hence:} \\
(\text{ii}) \quad \mu \varphi &= [t, a]\varphi = [t\mu^t, a\mu^s] = [t, a][t, \mu^s] = [t, \mu^s][t, a] = \mu^{(1-t)s_a+1}.
\end{align*}\]

Similarly, if
\[
\varphi^{-1} : a \mapsto a\mu^{s_a},
\]
then $\mu \varphi^{-1} = \mu^{(1-t)s_a+1}$. Thus,
\[
a(\varphi^{-1}\varphi) = (a\mu^{s_a})\varphi = a\mu^{s_a}(1-t)s_a = a.
\]

Consequently,
\[
r_a + [(1-t)r_a + 1]s_a = 0.
\]

Therefore $r_a$ and $s_a$ are mutual divisors, so they are associates in $\mathbb{Z}[t, t^{-1}]$. As a result, $(1-t)r_a + 1$ is a unit in $\mathbb{Z}[t, t^{-1}]$. The units of $\mathbb{Z}[t, t^{-1}]$ are of the form $\pm t^\gamma$.

If $1 + r_a = r_a t = t^\gamma$, then $r_a(t-1) = t^\gamma + 1$. This implies that $1$ is a root of $t^\gamma + 1$, a contradiction.

Hence $1 + r_a - r_a t = t^\gamma$, and $r_a(1-t) = t^\gamma - 1$. Thus,
\[
r_a = \begin{cases} 
-(1 + t + \cdots + t^{\gamma-1}) & \text{if } \gamma > 0, \\
t^{-1} + \cdots + t^{\gamma+1} + t^\gamma & \text{if } \gamma < 0.
\end{cases}
\]

Finally, suppose $r_1 = n_1t^{i_1} + n_2t^{i_2} + \cdots + n_kt^{i_k}$, where $n_j, i_j \in \mathbb{Z}$. A straightforward computation shows that $\varphi$ is the inner automorphism induced by $w = t^\gamma a_i^{n_1} \cdots a_{i_k}^{n_k} \in W$. \hfill \Box

**Example 2.** The following example is given by P. Hall in [6]. Let $B$ be the free nilpotent group of class 2, freely generated by $\{b_i : i \in \mathbb{Z}\}$. Thus
\[
B = \langle \ldots, b_{-1}, b_0, b_1, \ldots ; [b_i, b_j, b_k] \mid (i, j, k \in \mathbb{Z}) \rangle.
\]

Consider the quotient group
\[
N = \langle B ; [b_i, b_j] = [b_{i+1}, b_{j+1}] \mid (i, j \in \mathbb{Z}) \rangle.
\]

For each positive integer $r$, set $d_r = [b_0, b_r]$. In $N$, $d_r = [b_i, b_{i+r}]$ for all $i \in \mathbb{Z}$. In fact, the effect of the relations $[b_i, b_j] = [b_{i+1}, b_{j+1}]$ is that $N'$ is free abelian on $\{d_1, d_2, \ldots \}$.

Let
\[
a : b_i \mapsto b_{i+1}
\]
be the translation map on the generators of \( N \). By the nature of the defining relations of \( N \), \( a \) induces an automorphism of \( N \). Set \( b = b_0 \) and form the semi-direct product

\[ G = N \rtimes \langle a \rangle = gp(b, a) . \]

It is straightforward to verify that \( \zeta(G) = N' \). In particular, \( \zeta(G) \) is a free abelian group of countably infinite rank, contained in \( G' \). As a consequence of the defining relations for \( G \), it is the case that

\[ G/\zeta(G) \cong Z \wr Z . \]

Since \( Z \wr Z \) has trivial center, \( G \) is a two-generator group satisfying that

\[ \zeta(G/\zeta(G)) = 1 . \]

Thus, \( G \) is a two-generator \( H \)-group with \( Aut_c(G) = Aut_z(G) \), which can also be viewed as a matrix group (see [3], page 45, for details). Since \( \zeta(G) \) is not finitely generated, it follows from Lemma 4.2 that \( Aut_c(G) \) is not finitely generated. By Lemma 4.3 and Theorem 3.2 we conclude that the \( IA \)-group of \( G \) is not finitely generated.

Now, both Examples 1 and 2 lead to a collection of continuously many finitely generated center-by-metabelian groups whose \( IA \)-group is not finitely generated. Suppose \( G \) is either of the groups discussed in the above examples. Then \( \zeta(G) \) is free abelian of countably infinite rank. Let \( X \) be a free generating set for \( \zeta(G) \). It is known that there are continuously many subgroups \( G_{\sigma} \) of \( \zeta(G) \), each generated by a non-empty subset \( A_{\sigma} \) of \( X \), and that the quotient groups \( G/G_{\sigma} \) fall into continuously many isomorphism classes ([3], page 3). For each \( G_{\sigma} \),

\[ \zeta(G/G_{\sigma}) = \{ x \in G/G_{\sigma} : [x, g] \in G_{\sigma} (g \in G) \} . \]

Since \( G \) is an \( H \)-group and \( G_{\sigma} \leq \zeta(G) \), we conclude that

\[ \zeta(G/G_{\sigma}) \cong G/G_{\sigma}/\zeta(G/G_{\sigma}) . \]

By (4.1), \( \zeta(G/G_{\sigma}) \) is finitely generated only if the complement of \( A_{\sigma} \) is finite. Discard the subsets of \( X \) whose complement is finite. A continuous family of subsets \( A_{\sigma} \subset X \) remains, the corresponding quotient groups \( G/G_{\sigma} \) fall into continuously many isomorphism classes, and each \( \zeta(G/G_{\sigma}) \) is free abelian of countably infinite rank.

It follows again from (4.1) that \( \zeta(G/G_{\sigma}) \leq (G/G_{\sigma})' \), and

\[ G/G_{\sigma}/\zeta(G/G_{\sigma}) = G/G_{\sigma}/\zeta(G)/G_{\sigma} \cong G/\zeta(G) . \]

By Theorem 3.2 each \( G/G_{\sigma} \) is a 2-generator center-by-metabelian group whose \( IA \)-group is not finitely generated.

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