Non-Euclidean Geometry, Topology, and Networks

The Elements of Euclid is quite possibly the most influential mathematics book ever written. (See the margin note at the beginning of this chapter on page 492.) It begins with definitions of basic ideas such as point, line, and plane. Euclid then gives five postulates providing the foundation of all that follows.

Next, Euclid lists five axioms that he views as general truths and not just facts about geometry. (To some of the Greek writers, postulates were truths about a particular field, while axioms were general truths. Today, “axiom” is used in either case.)

Using only these ten statements and the basic rules of logic, Euclid was able to prove a large number of “propositions” about geometric figures.

The statement for Postulate 5 given above is actually known as Playfair’s axiom on parallel lines, which is equivalent to Euclid’s fifth postulate. To understand why this postulate caused trouble for so many mathematicians for so long, we must examine the original formulation.

Euclid’s Fifth Postulate (“Parallel Postulate”) Euclid’s fifth postulate states that if two lines \(k\) and \(m\) in Figure 74 are such that a third line, \(n\), intersects them so that the sum of the two interior angles \((A\) and \(B)\) on one side of line \(n\) is less than two right angles, then the two lines, if extended far enough, will meet on the same side of \(n\) that has the sum of the interior angles less than two right angles.

Euclid’s parallel postulate is quite different from the other nine postulates and axioms we listed. The others are simple statements that seem in complete agreement with our experience of the world around us. But the parallel postulate is long and wordy, and difficult to understand without a sketch.

The difference between the parallel postulate and the other axioms was noted by the Greeks, as well as later mathematicians. It was commonly believed that this was not a postulate at all, but a theorem to be proved. For more than 2000 years mathematicians tried repeatedly to prove it.

The most dedicated attempt came from an Italian Jesuit, Girolamo Saccheri (1667–1733). He attempted to prove the parallel postulate in an indirect way, by so-called “reduction to absurdity.” He would assume the postulate to be false and then show that the assumption leads to a contradiction of something true (an absurdity). Such a contradiction would thus prove the statement true.
Saccheri began with a quadrilateral, as in Figure 75. He assumed angles $A$ and $B$ to be right angles and sides $AD$ and $BC$ to be equal. His plan was as follows:

1. To assume that angles $C$ and $D$ are obtuse angles, and to show that this leads to a contradiction.
2. To assume that angles $C$ and $D$ are acute angles, and to show that this also leads to a contradiction.
3. Then if $C$ and $D$ can be neither acute nor obtuse angles, they must be right angles.
4. If $C$ and $D$ are both right angles, then it can be proved that the fifth postulate is true. It thus is a theorem rather than a postulate.

Saccheri had no trouble with part 1. However, he did not actually reach a contradiction in the second part, but produced some theorems so “repugnant” that he convinced himself he had vindicated Euclid. In fact, he published a book called in English *Euclid Freed of Every Flaw*.

Today we know that the fifth postulate is indeed an axiom, and not a theorem. It is consistent with Euclid’s other axioms.

The ten axioms of Euclid describe the world around us with remarkable accuracy. We now realize that the fifth postulate is necessary in Euclidean geometry to establish flatness. That is, the axioms of Euclid describe the geometry of plane surfaces. By changing the fifth postulate, we can describe the geometry of other surfaces. So, other geometric systems exist as much as Euclidean geometry exists, and they can even be demonstrated in our world. They are just not as familiar. A system of geometry in which the fifth postulate is changed is called a non-Euclidean geometry.

**The Origins of Non-Euclidean Geometry**

One non-Euclidean system was developed by three people working separately at about the same time. Early in the nineteenth century Carl Friedrich Gauss, one of the great mathematicians, worked out a consistent geometry replacing Euclid’s fifth postulate. He never published his work, however, because he feared the ridicule of people who could not free themselves from habitual ways of thinking. Gauss first used the term “non-Euclidean.” Nikolai Ivanovich Lobachevski (1793–1856) published a similar system in 1830 in the Russian language. At the same time, Janos Bolyai (1802–1860), a Hungarian army officer, worked out a similar system, which he published in 1832, not knowing about Lobachevski’s work. Bolyai never recovered from the disappointment of not being the first, and did no further work in mathematics.

Lobachevski replaced Euclid’s fifth postulate with:

Angles $C$ and $D$ in the quadrilateral of Saccheri are acute angles.

This postulate of Lobachevski can be rephrased as follows:

Through a point $P$ off a line $k$ (Figure 76), at least two different lines can be drawn parallel to $k$.

Compare this form of Lobachevski’s postulate to the geometry of Euclid. How many lines can be drawn through $P$ and parallel to $k$ in Euclidean geometry? At first glance, the postulate of Lobachevski does not agree with what we know about the world around us. But this is only because we think of our immediate surroundings as being flat.

Many of the theorems of Euclidean geometry are valid for the geometry of Lobachevski, but many are not. For example, in Euclidean geometry, the sum of the
measures of the angles in any triangle is $180^\circ$. In Lobachevskian geometry, the sum
of the measures of the angles in any triangle is less than $180^\circ$. Also, triangles of dif-
fferent sizes can never have equal angles, so similar triangles do not exist.

The geometry of Euclid can be represented on a plane. Since any portion of the
earth that we are likely to see looks flat, Euclidean geometry is very useful for de-
scribing the everyday world around us. The non-Euclidean geometry of Lobachevski
can be represented as a surface called a pseudosphere. This surface is formed by re-
volving a curve called a tractrix about the line $AB$ in Figure 77.

A second non-Euclidean system was developed by Georg Riemann (1826–1866).
He pointed out the difference between a line that continues indefinitely and a line
having infinite length. For example, a circle on the surface of a sphere continues

**FOR FURTHER THOUGHT**

Artists must have a sound knowledge of
gometry in order to paint realistically a world
of solids in space. Our three-dimensional world
must be represented in a convincing way on
canvas or some other plane surface. For
example, how should an artist paint a realistic
view of railroad tracks going off to the horizon?
In reality, the tracks are always at a constant
distance apart, but they cannot be drawn that
way except from overhead. The artist must
make the tracks converge at a point. Only in
this way will the scene look “real.”

Beginning in the fifteenth century, artists led
by Leone Battista Alberti, Leonardo da Vinci,
and Albrecht Dürer began to study the
problems of representing three dimensions in
two. They found geometric methods of doing
this. What artists initiated, mathematicians
developed into a geometry different from that
of Euclid—projective geometry.

Gerard Desargues (1591–1661), architect and
engineer, was a technical advisor to the French
government. He met Descartes at the siege of
La Rochelle in 1628 and in the 1630s was a
member of the Parisian group that included
Descartes, Pascal, Fermat, and Mersenne. In
1636 and 1639 he published a treatise and
proposals about the perspective section—
Desargues had invented projective geometry.
His works unfortunately were too difficult for
the times; his terms were from botany and he
did not use Cartesian symbolism. Desargues’s
geometric innovations were hidden for nearly

200 years. A manuscript by Desargues turned
up in 1845, about 30 years after Jean-Victor
Poncelet had rediscovered projective geometry.

**For Group Discussion**

Have a class member go to the board with a
long straightedge and draw a figure similar to
the one shown here.

Then have the student, very carefully, follow
these instructions.

1. Extend lines $AC$ and $A’C’$ of the two
   triangles to meet in a point $M$.
2. Extend sides $AB$ and $A’B’$ to meet in a point
   $N$.
3. Extend $BC$ and $B’C’$ to meet in a point $P$.

Now, as a class, make a conjecture about what
has happened. This exercise is an illustration of
Desargues’s theorem (stated in Exercise 12 of
this section).
Non-Euclidean Geometry, Topology, and Networks

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indefinitely but does not have infinite length. Riemann developed the idea of geometry on a sphere and replaced Euclid’s fifth postulate with:

Angles $C$ and $D$ of the Saccheri quadrilateral are obtuse angles.

In terms of parallel lines, Riemann’s postulate becomes:

Through a point $P$ off a line $k$, no line can be drawn that is parallel to $k$.

Riemannian geometry is important in navigation. “Lines” in this geometry are really great circles, or circles whose centers are at the center of the sphere. The shortest distance between two points on a sphere lies along an arc of a great circle. Great circle routes on a globe don’t look at all like the shortest distance when the globe is flattened out to form a map, but this is part of the distortion that occurs when the earth is represented as a flat surface. See Figure 78. The sides of a triangle drawn on a sphere would be arcs of great circles. And, in Riemannian geometry, the sum of the measures of the angles in any triangle is more than 180°.

Topology

This chapter began by suggesting that Euclidean geometry might seem rigid from a modern-day point of view. The plane and space figures studied in the Euclidean system are carefully distinguished by differences in size, shape, angularity, and so on. For a given figure such properties are permanent, and thus we can ask sensible questions about congruence and similarity. Suppose we studied “figures” made of rubber bands, as it were, “figures” that could be stretched, bent, or otherwise distorted without tearing or scattering. Topology, an important twentieth-century geometry, does just that.

Topological questions concern the basic structure of objects rather than size or arrangement. For example, a typical topological question has to do with the number of holes in an object, a basic structural property that does not change during deformation. You cannot deform a rubber ball to get a rubber band without tearing it—making a hole in it. Thus the two objects are not topologically equivalent. On the other hand, a doughnut and a coffee cup are topologically equivalent, since one could be stretched so as to form the other, without changing the basic structural property.

**EXAMPLE 1** Decide if the figures in each pair are topologically equivalent.

(a) a football and a cereal box

If we assume that a football is made of a perfectly elastic substance such as rubber or dough, it could be twisted or kneaded into the same shape as a cereal box. Thus, the two figures are topologically equivalent.

(b) a doughnut and an unzipped coat

A doughnut has one hole, while the coat has two (the sleeve openings). Thus, a doughnut could not be stretched and twisted into the shape of the coat without tearing another hole in it. Because of this, a doughnut and the coat are not topologically equivalent.

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**Topology in Medicine**

Trypanosome is a parasite that causes sleeping sickness in humans. The genetic material for this parasite contains a large number of linked circles. If the drug ethidium bromide is introduced into the cells, these circles become hopelessly intertwined, and the parasite cannot reproduce. Thus, topology can be used to cure sleeping sickness!
In topology, figures are classified according to their genus—that is, the number of cuts that can be made without cutting the figures into two pieces. The genus of an object is the number of holes in it. See Figure 79 on the next page.

Two examples of topological surfaces are the Möbius strip and the Klein bottle. The Möbius strip is a single-sided surface named after August Ferdinand Möbius (1790–1868), a pupil of Gauss.

To construct a Möbius strip, cut out a rectangular strip of paper, perhaps 3 cm by 25 cm. Paste together the two 3-cm ends after giving the paper a half-twist. To see how the strip now has only one side, mark an x on the strip and then mark another x on what appears to be the other “side.” Begin at one of the x’s you have drawn, and trace a path along the strip. You will eventually come to the other x without crossing the edge of the strip.

A branch of chemistry called chemical topology studies the structures of chemical configurations. A recent advance in this area was the synthesis of the first molecular Möbius strip, which was formed by joining the ends of a double-stranded strip of carbon and oxygen atoms.

A mathematician confided
That a Möbius strip is one-sided.
And you’ll get quite a laugh
If you cut one in half,
For it stays in one piece when divided.

Whereas a Möbius strip results from giving a paper strip a half-twist and then connecting it to itself, if we could do the same thing with a paper tube we would obtain a Klein bottle, named after Felix Klein (1849–1925). Klein produced important results in several areas, including non-Euclidean geometry and the early beginnings of group theory. (It is not possible to construct an actual Klein bottle. However, for some interesting blown glass models, see www.kleinbottle.com.)

A mathematician named Klein
Thought the Möbius strip was divine.
Said he, “If you glue
The edges of two
You’ll get a weird bottle like mine.”

For Group Discussion

1. The Möbius strip has other interesting properties. With a pair of scissors, cut the strip lengthwise. Do you get two strips? Repeat the process with what you have obtained from the first cut. What happens?
2. Now construct another Möbius strip, and start cutting lengthwise about 1/3 of the way from one edge. What happens?
3. What would be the advantage of a conveyor belt with the configuration of a Möbius strip?
9.7 Non-Euclidean Geometry, Topology, and Networks

Another branch of modern geometry is graph theory. One topic of study in graph theory is networks. A network is a diagram showing the various paths (or arcs) between points (called vertices, or nodes). A network can be thought of as a set of arcs and vertices. Figure 80 shows two examples of networks. The study of networks began formally with the so-called Königsberg Bridge problem as solved by Leonhard Euler (1707–1783). In Königsberg, Germany, the River Pregel flowed through the middle of town. There were two islands in the river. During Euler's lifetime, there were seven bridges connecting the islands and the two banks of the river.

The people of the town loved Sunday strolls among the various bridges. Gradually, a competition developed to see if anyone could find a route that crossed each of the seven bridges exactly once. The problem concerns what topologists today call the traversability of a network. No one could find a solution. The problem became so famous that in 1735 it reached Euler, who was then at the court of the Russian empress Catherine the Great. In trying to solve the problem, Euler began by drawing a network representing the system of bridges, as in Figure 81.

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*An entire chapter on Graph Theory, of which networks is one topic, is available from the publisher. If this chapter is not part of this book and you are interested in more information on networks, contact Addison-Wesley.
Euler first noticed that three routes meet at vertex $A$. Since 3 is an odd number, he called $A$ an **odd vertex**. Since three routes meet at $A$, it must be a starting or an ending point for any traverse of the network. This must be true; otherwise, when you got to $A$ on your second trip there would be no way to get out. An **even vertex**, one where an even number of routes meet, need not be a starting or an ending point. (Why is this?) Three paths also meet at $C$ and $D$, with five paths meeting at $B$. Thus, $B$, $C$, and $D$ are also odd vertices. An odd vertex must be a starting or an ending point of a traverse. Thus, all four vertices $A$, $B$, $C$, and $D$ must be starting or ending points. Since a network can have only two starting or ending points (one of each), this network cannot be traversed. The residents of Königsberg were trying to do the impossible.

Euler’s result can be summarized as follows.

### Results on Vertices and Traversability

1. The number of odd vertices of any network is even. (That is, a network must have $2n$ odd vertices, where $n = 0, 1, 2, 3, \ldots$)
2. A network with no odd vertices or exactly two odd vertices can be traversed. In the case of exactly two, start at one odd vertex and end at the other.
3. A network with more than two odd vertices cannot be traversed.

#### EXAMPLE 2
Decide whether the networks in Figure 82 and Figure 83 are traversable.

Since there are exactly two odd vertices ($A$ and $E$) in Figure 82, it can be traversed. One way to traverse the network of the figure is to start at $A$, go through $B$ to $C$, then back to $A$. (It is acceptable to go through a vertex as many times as needed.) Then go to $E$, to $D$, to $C$, and finally go back to $E$. It is traversable.

In Figure 83, because vertices $A$, $B$, $C$, and $D$ are all odd (five routes meet at each of them), this network is not traversable. (One of the authors of this text, while in high school, tried for hours to traverse it, not knowing he was attempting the impossible!)

#### EXAMPLE 3
Figure 84 shows the floor plan of a house. Is it possible to travel through this house, going through each door exactly once?

Rooms $A$, $C$, and $D$ have even numbers of doors, while rooms $B$ and $E$ have odd numbers of doors. If we think of the rooms as the vertices of a graph, then the fact that we have exactly two odd vertices means that it is possible to travel through each door of the house exactly once. (Show how—start in room $B$ or room $E$.)

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**FIGURE 82**

**FIGURE 83**

**FIGURE 84**

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