Exercises 89 and 90 refer to the given figure. The center of the circle is O.

89. Radius of a Circle If $\overrightarrow{AC}$ measures 6 in. and $\overrightarrow{BC}$ measures 8 in., what is the radius of the circle?

90. Lengths of Chords of a Circle If $\overrightarrow{AB}$ measures 13 cm, and the length of $\overrightarrow{BC}$ is 7 cm more than the length of $\overrightarrow{AC}$, what are the lengths of $\overrightarrow{BC}$ and $\overrightarrow{AC}$?

Verify that the following constructions from the Extension following Section 9.2 are valid. Use a STATEMENTS/REASONS proof.

91. Construction 1
92. Construction 2
93. Construction 3
94. Construction 4

### 9.5 Space Figures, Volume, and Surface Area

**Space Figures** Thus far, this chapter has discussed only plane figures—figures that can be drawn completely in the plane of a piece of paper. However, it takes the three dimensions of space to represent the solid world around us. For example, Figure 52 shows a “box” (a rectangular parallelepiped in mathematical terminology). The faces of a box are rectangles. The faces meet at edges; the “corners” are called vertices (plural of vertex—the same word as for the “corner” of an angle).

Boxes are one kind of space figure belonging to an important group called polyhedrons, the faces of which are made only of polygons. Perhaps the most interesting polyhedrons are the regular polyhedrons. Recall that a regular polygon is a polygon with all sides equal and all angles equal. A regular polyhedron is a space figure, the faces of which are only one kind of regular polygon. It turns out that there are only five different regular polyhedrons. They are shown in Figure 53. A tetrahedron is composed of four equilateral triangles, each three of which meet in a point. Use the figure to verify that there are four faces, four vertices, and six edges.

![Figure 52: Box with labels for vertex, face, and edge.](image)

![Figure 53: Regular polyhedrons.](image)

A regular quadrilateral is called a square. Six squares, each three of which meet at a point, form a hexahedron, or cube. Again, use Figure 53 to verify that a cube has 6 faces, 8 vertices, and 12 edges.

The three remaining regular polyhedrons are the octahedron, the dodecahedron, and the icosahedron. The octahedron is composed of groups of four regular triangles (i.e., equilateral) meeting at a point. The dodecahedron is formed by groups of three regular pentagons, while the icosahedron is made up of groups of five regular triangles.
Two other types of polyhedrons are familiar space figures: pyramids and prisms. **Pyramids** are made of triangular sides and a polygonal base. **Prisms** have two faces in parallel planes; these faces are congruent polygons. The remaining faces of a prism are all parallelograms. (See Figure 54(a) and (b) on the next page.) By this definition, a box is also a prism.
The circle, although a plane figure, is not a polygon. (Why?) Figure 54(c) shows space figures made up in part of circles, including right circular cones and right circular cylinders. The figure also shows how a circle can generate a torus, a doughnut-shaped solid that has interesting topological properties. See Section 9.7.

Volume and Surface Area

While area is a measure of surface covered by a plane figure, volume is a measure of capacity of a space figure. Volume is measured in cubic units. For example, a cube with edge measuring 1 cm has volume 1 cubic cm, which is also written as 1 cm$^3$, or 1 cc. The surface area is the total area that would be covered if the space figure were “peeled” and the peel laid flat. Surface area is measured in square units.

Volume and Surface Area of a Box

Suppose that a box has length $\ell$, width $w$, and height $h$. Then the volume $V$ and the surface area $S$ are given by the formulas

$$V = \ell wh$$

and

$$S = 2\ell w + 2\ell h + 2hw.$$

In particular, if the box is a cube with edge of length $s$,

$$V = s^3$$

and

$$S = 6s^2.$$
Example 1

Find the volume \( V \) and the surface area \( S \) of the box shown in Figure 55.

**FIGURE 55**

To find the volume, use the formula \( V = \ell w h \) with \( \ell = 14 \), \( w = 7 \), and \( h = 5 \).

\[
V = \ell w h = 14 \cdot 7 \cdot 5 = 490
\]

Volume is measured in cubic units, so the volume of the box is 490 cubic centimeters, or \( 490 \text{ cm}^3 \).

To find the surface area, use the formula \( S = 2\ell w + 2\ell h + 2hw \).

\[
S = 2(14)7 + 2(14)5 + 2(5)7 = 196 + 140 + 70 = 406
\]

Like areas of plane figures, surface areas of space figures are given in square measure, so the surface area of the box is 406 square centimeters, or \( 406 \text{ cm}^2 \).

A typical tin can is an example of a right circular cylinder.

---

**Example 2**

In Figure 56, the right circular cylinder has surface area \( 288\pi \) square inches, and the radius of its base is 6 inches.

(a) Find the height of the cylinder.

Since we know that \( S = 288\pi \) and \( r = 6 \), substitute into the formula for surface area to find \( h \).

\[
S = 2\pi rh + 2\pi r^2
\]

\[
288\pi = 2\pi(6)h + 2\pi(6)^2
\]

\[
288\pi = 12\pi h + 72\pi
\]

\[
216\pi = 12\pi h
\]

\[
h = 18
\]

The height is 18 inches.
(b) Find the volume of the cylinder.

Use the formula for volume, with \( r = 6 \) and \( h = 18 \).

\[
V = \pi r^2 h = \pi (6)^2 (18) = 648 \pi
\]

The exact volume is \( 648\pi \) cubic inches, or approximately 2034.72 cubic inches, using \( \pi = 3.14 \).

The three-dimensional analogue of a circle is a sphere. It is defined by replacing the word “plane” with “space” in the definition of a circle (Section 9.2).

**EXAMPLE 3** Suppose that a spherical tank having radius 3 meters can be filled with liquid fuel for $200. How much will it cost to fill a spherical tank of radius 6 meters with the same fuel?

We must first find the volume of the tank with radius 3 meters. Call it \( V_1 \).

\[
V = \frac{4}{3} \pi r^3
\]

\[
V_1 = \frac{4}{3} \pi (3)^3 = \frac{4}{3} \pi (27) = 36 \pi
\]

Now find \( V_2 \), the volume of the tank having radius 6 meters.

\[
V_2 = \frac{4}{3} \pi (6)^3 = \frac{4}{3} \pi (216) = 288 \pi
\]

Notice that by doubling the radius of the sphere from 3 meters to 6 meters, the volume has increased 8 times, since

\[
V_2 = 288 \pi = 8V_1 = 8(36 \pi).
\]

Therefore, the cost to fill the larger tank is eight times the cost to fill the smaller one: \( 8($200) = $1600. \)

The space figure shown in Figure 57 is a right circular cone.
Volume and Surface Area of a Right Circular Cone

If a right circular cone has height \( h \) and the radius of its circular base is \( r \), then the volume \( V \) and the surface area \( S \) are given by the formulas

\[
V = \frac{1}{3} \pi r^2 h
\]

and

\[
S = \pi r \sqrt{r^2 + h^2} + \pi r^2.
\]

(In the formula for \( S \), the area of the bottom is included.)

A pyramid is a space figure having a polygonal base and triangular sides. Figure 58 shows a pyramid with a square base.

Volume of a Pyramid

If \( B \) represents the area of the base of a pyramid, and \( h \) represents the height (that is, the perpendicular distance from the top, or apex, to the base), then the volume \( V \) is given by the formula

\[
V = \frac{1}{3} Bh.
\]

**Example 4** What is the ratio of the volume of a right circular cone with radius of base \( r \) and height \( h \) to the volume of a pyramid having a square base, with each side of length \( r \), and height \( h \)?

Using the formula for the volume of a cone, we have

\[
V_1 = \text{Volume of the cone} = \frac{1}{3} \pi r^2 h.
\]

Since the pyramid has a square base, the area \( B \) of its base is \( r^2 \). Using the formula for the volume of a pyramid, we get

\[
V_2 = \text{Volume of the pyramid} = \frac{1}{3} Bh = \frac{1}{3} (r^2)h.
\]

The ratio of the first volume to the second is

\[
\frac{V_1}{V_2} = \frac{\frac{1}{3} \pi r^2 h}{\frac{1}{3} r^2 h} = \pi.
\]