75. **Area of a Square** The area of square $PQRS$ is 1250 square feet. $T$, $U$, $V$, and $W$ are the midpoints of $PQ$, $QR$, $RS$, and $SP$, respectively. What is the area of square $TUVW$?

![Diagram of a square with midpoints](image1)

76. **Area of a Quadrilateral** The rectangle $ABCD$ has length twice the width. If $P$, $Q$, $R$, and $S$ are the midpoints of the sides, and the perimeter of $ABCD$ is 96 in., what is the area of quadrilateral $PQRS$?

![Diagram of a rectangle with midpoints](image2)

77. **Area of a Shaded Region** If $ABCD$ is a square with each side measuring 36 in., what is the area of the shaded region?

![Diagram of a square with a shaded region](image3)

78. **Perimeter of a Polygon** Can the perimeter of the polygon shown be determined from the given information? If so, what is the perimeter?

![Diagram of a polygon](image4)

79. **Area of a Shaded Region** Express the area of the shaded region in terms of $r$, given that the circle is inscribed in the square.

![Diagram of a circle inscribed in a square](image5)

80. **Area of a Trapezoid** Find the area of trapezoid $ABCD$, given that the area of right triangle $ABE$ is $30 \text{ in.}^2$.

![Diagram of a trapezoid](image6)

---

**The Geometry of Triangles: Congruence, Similarity, and the Pythagorean Theorem**

**Congruent Triangles** In this section we investigate special properties of triangles. Triangles that are both the same size and the same shape are called **congruent triangles**. Informally speaking, if two triangles are congruent, then it is possible to pick up one of them and place it on top of the other so that they coincide exactly. An everyday example of congruent triangles would be the triangular supports for a child’s swing set, machine-produced with exactly the same dimensions each time.

Figure 39 illustrates two congruent triangles, $\triangle ABC$ and $\triangle DEF$. The symbol $\cong$ denotes congruence, so $\triangle ABC \cong \triangle DEF$. Notice how the angles and sides are marked to indicate which angles are congruent and which sides are congruent. (Using precise terminology, we refer to angles or sides as being **congruent**, while...
the measures of congruent angles or congruent sides are equal. We will often use the terms “equal angles” or “equal sides” to describe angles of equal measure or sides of equal measure.)

In geometry the following properties are used to prove that two triangles are congruent.

**Congruence Properties**

**Side-Angle-Side (SAS)** If two sides and the included angle of one triangle are equal, respectively, to two sides and the included angle of a second triangle, then the triangles are congruent.

**Angle-Side-Angle (ASA)** If two angles and the included side of one triangle are equal, respectively, to two angles and the included side of a second triangle, then the triangles are congruent.

**Side-Side-Side (SSS)** If three sides of one triangle are equal, respectively, to three sides of a second triangle, then the triangles are congruent.

Examples 1–3 show how to prove statements using these properties. We use a diagram with two columns, headed by STATEMENTS and REASONS.

---

**EXAMPLE 1** Refer to Figure 40.

**Given:** \( CE = ED \)
\( AE = EB \)

**Prove:** \( \triangle ACE \cong \triangle BDE \)

**Proof:**

<table>
<thead>
<tr>
<th>STATEMENTS</th>
<th>REASONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( CE = ED )</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( AE = EB )</td>
<td>2. Given</td>
</tr>
<tr>
<td>3. ( \angle CEA = \angle DED )</td>
<td>3. Vertical angles are equal.</td>
</tr>
<tr>
<td>4. ( \triangle ACE \cong \triangle BDE )</td>
<td>4. SAS congruence property</td>
</tr>
</tbody>
</table>

**EXAMPLE 2** Refer to Figure 41.

**Given:** \( \angle ADB = \angle CBD \)
\( \angle ABD = \angle CDB \)

**Prove:** \( \triangle ADB \cong \triangle CBD \)

**Proof:**

<table>
<thead>
<tr>
<th>STATEMENTS</th>
<th>REASONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \angle ADB = \angle CBD )</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( \angle ABD = \angle CDB )</td>
<td>2. Given</td>
</tr>
<tr>
<td>3. ( DB = DB )</td>
<td>3. Reflexive property (a quantity is equal to itself)</td>
</tr>
<tr>
<td>4. ( \triangle ADB \cong \triangle CBD )</td>
<td>4. ASA congruence property</td>
</tr>
</tbody>
</table>

---

**Plimpton 322** Our knowledge of the mathematics of the Babylonians of Mesopotamia is based largely on archaeological discoveries of thousands of clay tablets. On the tablet labeled Plimpton 322, there are several columns of inscriptions that represent numbers. The far right column is simply one that serves to number the lines, but two other columns represent values of hypotenuses and legs of right triangles with integer-valued sides. Thus, it seems that while the famous theorem relating right-triangle side lengths is named for the Greek Pythagoras, the relationship was known more than 1000 years prior to the time of Pythagoras.
EXAMPLE 3  Refer to Figure 42.

Given:  \( AD = CD \)
\( AB = CB \)

Prove:  \( \triangle ABD \equiv \triangle CBD \)

Proof:

<table>
<thead>
<tr>
<th>STATEMENTS</th>
<th>REASONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( AD = CD )</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( AB = CB )</td>
<td>2. Given</td>
</tr>
<tr>
<td>3. ( BD = BD )</td>
<td>3. Reflexive property</td>
</tr>
<tr>
<td>4. ( \triangle ABD \equiv \triangle CBD )</td>
<td>4. SSS congruence property</td>
</tr>
</tbody>
</table>

In Example 3, \( \triangle ABC \) is an isosceles triangle. The results of that example allow us to make several important statements about an isosceles triangle. They are indicated symbolically in Figure 43 and stated in the following box.

Important Statements About Isosceles Triangles

If \( \triangle ABC \) is an isosceles triangle with \( AB = CB \), and if \( D \) is the midpoint of the base \( AC \), then the following properties hold.

1. The base angles \( A \) and \( C \) are equal.
2. Angles \( ABD \) and \( CBD \) are equal.
3. Angles \( ADB \) and \( CDB \) are both right angles.

Similar Triangles  Many of the key ideas of geometry depend on similar triangles, pairs of triangles that are exactly the same shape but not necessarily the same size. Figure 44 shows three pairs of similar triangles. (Note: The triangles do not need to be oriented in the same fashion in order to be similar.)

Suppose that a correspondence between two triangles \( ABC \) and \( DEF \) is set up as follows.

\[ A \text{ corresponds to } D \]
\[ B \text{ corresponds to } E \]
\[ C \text{ corresponds to } F \]

For triangle \( ABC \) to be similar to triangle \( DEF \), the following conditions must hold.

1. Corresponding angles must have the same measure.
2. The ratios of the corresponding sides must be constant; that is, the corresponding sides are proportional.

By showing that either of these conditions holds in a pair of triangles, we may conclude that the triangles are similar.

EXAMPLE 4  In Figure 45, \( AB \) is parallel to \( ED \). How can we verify that \( \triangle ABC \) is similar to \( \triangle EDC \)?

Because \( AB \) is parallel to \( ED \), the transversal \( \overrightarrow{BD} \) forms equal alternate interior angles \( ABC \) and \( EDC \). Also, transversal \( \overrightarrow{AE} \) forms equal alternate interior angles \( BAC \) and \( DEC \). We know that \( \angle ACB = \angle ECD \), because they are vertical angles. Because the corresponding angles have the same measures in triangles \( ABC \) and \( EDC \), the triangles are similar.
Once we have shown that two angles of one triangle are equal to the two corresponding angles of a second triangle, it is not necessary to show the same for the third angle. Since, in any triangle, the sum of the angles equals $180^\circ$, we may conclude that the measures of the remaining angles must be equal. This leads to the following Angle-Angle similarity property.

### Angle-Angle (AA) Similarity Property

If the measures of two angles of one triangle are equal to those of two corresponding angles of a second triangle, then the two triangles are similar.

**EXAMPLE 5** In Figure 46, $\triangle EDF$ is similar to $\triangle CAB$. Find the unknown side lengths in $\triangle EDF$.

As mentioned above, similar triangles have corresponding sides in proportion. Use this fact to find the unknown sides in the smaller triangle. Side $DF$ of the small triangle corresponds to side $AB$ of the larger one, and sides $DE$ and $AC$ correspond. This leads to the proportion

\[
\frac{8}{16} = \frac{DF}{24}.
\]

Using a technique from algebra, set the two cross-products equal. (As an alternative method of solution, multiply both sides by 48, the least common multiple of 16 and 24.) Setting cross-products equal gives

\[
8(24) = 16DF
\]

\[
192 = 16DF
\]

\[
12 = DF.
\]

Side $DF$ has length 12.

Side $EF$ corresponds to side $CB$. This leads to another proportion.

\[
\frac{8}{16} = \frac{EF}{32}
\]

\[
1 = \frac{EF}{32}
\]

\[
2EF = 32
\]

\[
EF = 16.
\]

Side $EF$ has length 16.

**EXAMPLE 6** Find the measures of the unknown parts of the similar triangles $\triangle STU$ and $\triangle ZXY$ in Figure 47.

Here angles $X$ and $T$ correspond, as do angles $Y$ and $U$, and angles $Z$ and $S$. Since angles $Z$ and $S$ correspond and since angle $S$ is $52^\circ$, angle $Z$ also must be $52^\circ$. The sum of the angles of any triangle is $180^\circ$. In the larger triangle $X = 71^\circ$ and $Z = 52^\circ$. To find $Y$, set up an equation and solve for $Y$. 

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Photo not available
9.4 The Geometry of Triangles: Congruence, Similarity, and the Pythagorean Theorem

\[ X + Y + Z = 180 \]
\[ 71 + Y + 52 = 180 \]
\[ 123 + Y = 180 \]
\[ Y = 57 \]

Angle \( Y \) is 57°. Since angles \( Y \) and \( U \) correspond, \( U = 57° \) also.

Now find the unknown sides. Sides \( SU \) and \( ZY \) correspond, as do \( TS \) and \( XZ \), and \( TU \) and \( XY \), leading to the following proportions.

\[
\begin{align*}
\frac{SU}{ZY} &= \frac{TS}{XZ} & \frac{XY}{ZY} &= \frac{ZU}{SU} \\
48 &= \frac{TS}{XZ} & XY &= \frac{ZU}{SU} \\
144 &= \frac{XY}{ZY} & 144 &= \frac{XY}{ZY} \\
3 &= \frac{SU}{ZY} & 3 &= \frac{SU}{ZY} \\
126 &= \frac{TU}{XY} & 126 &= \frac{SU}{ZY} \\
10 &= \frac{XY}{ZY} & 10 &= \frac{XY}{ZY} \\
99 &= \frac{XY}{ZY} & 99 &= \frac{XY}{ZY} \\
18 &= \frac{XY}{ZY} & 18 &= \frac{XY}{ZY}
\end{align*}
\]

Side \( TS \) has length 42, and side \( XY \) has length 120.

**EXAMPLE 7** Lucie Wanersdorfer, the Lettsworth, LA postmaster, wants to measure the height of the office flagpole. She notices that at the instant when the shadow of the station is 18 feet long, the shadow of the flagpole is 99 feet long. The building is 10 feet high. What is the height of the flagpole?

Figure 48 shows the information given in the problem. The two triangles shown there are similar, so that corresponding sides are in proportion, with

\[
\begin{align*}
\frac{MN}{10} &= \frac{99}{18} \\
\frac{MN}{10} &= \frac{11}{2} \\
2MN &= 110 \\
MN &= 55.
\end{align*}
\]

The flagpole is 55 feet high.

**The Pythagorean Theorem** We have used the Pythagorean theorem earlier in this book, and because of its importance in mathematics, we will investigate it further in this section on the geometry of triangles. Recall that in a right triangle, the side opposite the right angle (and consequently, the longest side) is called the hypotenuse. The other two sides, which are perpendicular, are called the legs.

**Pythagorean Theorem**

If the two legs of a right triangle have lengths \( a \) and \( b \), and the hypotenuse has length \( c \), then

\[ a^2 + b^2 = c^2. \]

That is, the sum of the squares of the lengths of the legs is equal to the square of the hypotenuse.
Pythagoras did not actually discover the theorem that was named after him, although legend tells that he sacrificed 100 oxen to the gods in gratitude for the discovery. There is evidence that the Babylonians knew the concept quite well. The first proof, however, may have come from Pythagoras.

Figure 49 illustrates the theorem in a simple way, by using a sort of tile pattern. In the figure, the side of the square along the hypotenuse measures 5 units, while those along the legs measure 3 and 4 units. If we let $a = 3$, $b = 4$, and $c = 5$, we see that the equation of the Pythagorean theorem is satisfied.

$$a^2 + b^2 = c^2$$

$$3^2 + 4^2 = 5^2$$

$$9 + 16 = 25$$

$$25 = 25$$
The natural numbers 3, 4, 5 form a **Pythagorean triple** since they satisfy the equation of the Pythagorean theorem. There are infinitely many such triples.

**Example 8**  
Find the length $a$ in the right triangle shown in Figure 50.
Let $b = 36$ and $c = 39$. Substituting these values into the equation $a^2 + b^2 = c^2$ allows us to solve for $a$.

\[
\begin{align*}
  a^2 + b^2 &= c^2 \\
  a^2 + 36^2 &= 39^2 \\
  a^2 + 1296 &= 1521 \\
  a^2 &= 225 \\
  a &= 15
\end{align*}
\]

Verify that 15, 36, 39 is a Pythagorean triple as a check.

**Problem Solving**

The Cairo Mathematical Papyrus is an Egyptian document that dates back to about 300 B.C. It was discovered in 1938 and examined in 1962. It contains forty problems, and nine of these deal with the Pythagorean relationship. One of these problems is solved in Example 9.

**Example 9**  
A ladder of length 10 cubits has its foot 6 cubits from a wall.  
To what height does the ladder reach?

As suggested by Figure 51, the ladder forms the hypotenuse of a right triangle, and the ground and wall form the legs. Let $x$ represent the distance from the base of the wall to the top of the ladder. Then, by the Pythagorean theorem,

\[
\begin{align*}
  x^2 + 6^2 &= 10^2 \\
  x^2 + 36 &= 100 \\
  x^2 &= 64 \\
  x &= 8
\end{align*}
\]

The ladder reaches a height of 8 cubits.

The statement of the Pythagorean theorem is an *if . . . then* statement. If the antecedent (the statement following the word “if”) and the consequent (the statement following the word “then”) are interchanged, the new statement is called the **converse** of the original one. Although the converse of a true statement may not be true, the **converse** of the Pythagorean theorem **is** also a true statement and can be used to determine if a triangle is a right triangle, given the lengths of the three sides.

**Converse of the Pythagorean Theorem**

If a triangle has sides of lengths $a$, $b$, and $c$, where $c$ is the length of the longest side, and if $a^2 + b^2 = c^2$, then the triangle is a right triangle.
Problem Solving

If a carpenter is building a floor for a rectangular room, it is essential that the corners of the floor represent right angles; otherwise, problems will occur when the walls are constructed, when flooring is laid, and so on. To check that the floor is “squared off,” the carpenter can use the converse of the Pythagorean theorem.

**EXAMPLE 10** A carpenter has been contracted to add an 8-foot-by-12-foot laundry room onto an existing house. After the floor has been built, he finds that the length of the diagonal of the floor is 14 feet, 8 inches. Is the floor “squared off” properly?

Since 14 feet, 8 inches = 14 2/3 feet, he must check to see whether the following statement is true.

\[
8^2 + 12^2 = \left(14 \frac{2}{3}\right)^2 \quad ?
\]

\[
8^2 + 12^2 = \left(\frac{44}{3}\right)^2 \quad ?
\]

\[
208 = \frac{1936}{9} \quad ?
\]

\[
208 \neq 215 \frac{1}{9} \quad \text{False}
\]

He has a real problem now, since his diagonal, which measures 14 feet, 8 inches, should actually measure \(\sqrt{208} = 14.4 \approx 14\) feet, 5 inches. He must correct his error to avoid major problems later.

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