44. If the early Greeks knew the form of all even perfect numbers, namely \(2^{n-1}(2^n - 1)\), then why did they not discover all the ones that are known today?

45. Explain why the primorial formula \(p\# \pm 1\) does not result in a pair of twin primes for the prime value \(p = 2\).

46. (a) What two numbers does the primorial formula produce for \(p = 7\)?
   
   (b) Which, if either, of these numbers is prime?

47. Choose the correct completion: The primorial formula produces twin primes
   

48. Choose the correct completion: The primorial formula produces twin primes
   

49. Explain why the factorial prime formula does not give twin primes for \(n = 2\).

50. Based on the preceding table, complete each of the following statements with one of the following: A. never, B. sometimes, or C. always. When applied to particular values of \(n\), the factorial formula \(n! \pm 1\) produces

   58. no primes ______.

   59. exactly one prime ______.

   60. twin primes ______.

### 5.3 Greatest Common Factor and Least Common Multiple

The greatest common factor (GCF) of a group of natural numbers is the largest natural number that is a factor of all the numbers in the group. For example, 18 is the greatest common factor of 36 and 54, since 18 is the largest natural number that divides both 36 and 54. Also, 1 is the greatest common factor of 7 and 18.

Greatest common factors can be found by using prime factorizations. To find the GCF of 36 and 54, first write the prime factorization of each number:

\[
36 = 2^2 \cdot 3^2 \quad \text{and} \quad 54 = 2^1 \cdot 3^3.
\]
The GCF is the product of the primes common to the factorizations, with each prime raised to the power indicated by the smallest exponent that it has in any factorization. Here, the prime 2 has 1 as the smallest exponent (in $54 = 2^1 \cdot 3^3$), while the prime 3 has 2 as the smallest exponent (in $36 = 2^2 \cdot 3^2$). The GCF of 36 and 54 is

$$2^1 \cdot 3^2 = 2 \cdot 9 = 18,$$

as stated earlier. We summarize as follows.

### Finding the Greatest Common Factor (Prime Factors Method)

1. Write the prime factorization of each number.
2. Choose all primes common to all factorizations, with each prime raised to the smallest exponent that it has in any factorization.
3. Form the product of all the numbers in Step 2; this product is the greatest common factor.

**EXAMPLE 1** Find the greatest common factor of 360 and 2700.

Write the prime factorization of each number:

$$360 = 2^3 \cdot 3^2 \cdot 5$$  and  $$2700 = 2^2 \cdot 3^3 \cdot 5^2.$$ 

Now find the primes common to both factorizations, with each prime having as exponent the smallest exponent from either product: $2^2, 3^2, 5$. Then form the product of these numbers.

$$\text{GCF} = 2^2 \cdot 3^2 \cdot 5 = 180$$ 

The greatest common factor of 360 and 2700 is 180.

**EXAMPLE 2** Find the greatest common factor of 720, 1000, and 1800.

Write the prime factorization for each number:

$$720 = 2^4 \cdot 3^2 \cdot 5,$$  $$1000 = 2^3 \cdot 5^3,$$  and  $$1800 = 2^3 \cdot 3^2 \cdot 5^2.$$ 

Use the smallest exponent on each prime common to the factorizations:

$$\text{GCF} = 2^3 \cdot 5 = 40.$$ 

(The prime 3 is not used in the greatest common factor since it does not appear in the prime factorization of 1000.)

**EXAMPLE 3** Find the greatest common factor of 80 and 63.

Start with

$$80 = 2^4 \cdot 5$$  and  $$63 = 3^2 \cdot 7.$$ 

There are no primes in common here, so the GCF is 1. The number 1 is the largest number that will divide into both 80 and 63.

Two numbers, such as 80 and 63, with a greatest common factor of 1 are called relatively prime numbers—that is, they are prime relative to one another.
Another method of finding the greatest common factor involves dividing the numbers by common prime factors.

**Finding the Greatest Common Factor (Dividing by Prime Factors Method)**

1. Write the numbers in a row.
2. Divide each of the numbers by a common prime factor. Try 2, then try 3, and so on.
3. Divide the quotients by a common prime factor. Continue until no prime will divide into all the quotients.
4. The product of the primes in Steps 2 and 3 is the greatest common factor.

This method is illustrated in the next example.

**EXAMPLE 4** Find the greatest common factor of 12, 18, and 60.

Write the numbers in a row and divide by 2.

\[
\begin{array}{c|c|c|c}
2 & 12 & 18 & 60 \\
6 & 9 & 30
\end{array}
\]

The numbers 6, 9, and 30 are not all divisible by 2, but they are divisible by 3.

\[
\begin{array}{c|c|c|c}
2 & 12 & 18 & 60 \\
3 & 6 & 9 & 30 \\
2 & 3 & 10
\end{array}
\]

No prime divides into 2, 3, and 10, so the greatest common factor of the numbers 12, 18, and 60 is given by the product of the primes on the left, 2 and 3.

\[
\begin{array}{c|c|c|c}
2 & 12 & 18 & 60 \\
3 & 6 & 9 & 30 \\
2 & 3 & 10
\end{array}
\]

\[
2 \cdot 3 = 6
\]

The GCF of 12, 18, and 60 is 6.

There is yet another method of finding the greatest common factor of two numbers (but not more than two) that does not require factoring into primes or successively dividing by primes. It is called the **Euclidean algorithm**, a

*For a proof that this process does indeed give the greatest common factor, see *Elementary Introduction to Number Theory, Second Edition*, by Calvin T. Long, pp. 34–35.*
5.3 Greatest Common Factor and Least Common Multiple

**Example 5** Use the Euclidean algorithm to find the greatest common factor of 90 and 168.

**Step 1:** Begin by dividing the larger, 168, by the smaller, 90. Disregard the quotient, but note the remainder.

90 | 168  
---|---
    | 1  
90 | 78  
---|---
    | 2

**Step 2:** Divide the smaller of the two numbers by the remainder obtained in Step 1. Once again, note the remainder.

78 | 90  
---|---
    | 1  
78 | 12  
---|---
    | 6

**Step 3:** Continue dividing the successive remainders, as many times as necessary to obtain a remainder of 0.

72 | 78  
---|---
    | 6  
72 | 6  
---|---
    | 0

**Step 4:** The last positive remainder in this process is the greatest common factor of 90 and 168. It can be seen that their GCF is 6.

The Euclidean algorithm is particularly useful if the two numbers are difficult to factor into primes. We summarize the algorithm here.

**Finding the Greatest Common Factor (Euclidean Algorithm)**

To find the greatest common factor of two unequal numbers, divide the larger by the smaller. Note the remainder, and divide the previous divisor by this remainder. Continue the process until a remainder of 0 is obtained. The greatest common factor is the last positive remainder obtained in this process.

Closely related to the idea of the greatest common factor is the concept of the least common multiple. The least common multiple (LCM) of a group of natural numbers is the smallest natural number that is a multiple of all the numbers in the group. For example, if we wish to find the least common multiple of 15 and 10, we can specify the sets of multiples of 15 and multiples of 10.

Multiples of 15: \{15, 30, 45, 60, 75, 90, 105, \ldots\}

Multiples of 10: \{10, 20, 30, 40, 50, 60, 70, \ldots\}

The set of natural numbers that are multiples of both 15 and 10 form the set of common multiples:

\{30, 60, 90, 120, \ldots\}.

While there are infinitely many common multiples, the least common multiple is observed to be 30.
A method similar to the first one given for the greatest common factor may be used to find the least common multiple of a group of numbers.

**Finding the Least Common Multiple (Prime Factors Method)**

1. Write the prime factorization of each number.
2. Choose all primes belonging to *any* factorization, with each prime raised to the power indicated by the largest exponent that it has in any factorization.
3. Form the product of all the numbers in Step 2; this product is the least common multiple.

**EXAMPLE 6** Find the least common multiple of 135, 280, and 300.

Write the prime factorizations:

- $135 = 3^3 \cdot 5$
- $280 = 2^3 \cdot 5 \cdot 7$
- $300 = 2^2 \cdot 3 \cdot 5^2$

Form the product of all the primes that appear in *any* of the factorizations. Use the largest exponent from any factorization.

$$LCM = 2^3 \cdot 3^3 \cdot 5^2 \cdot 7 = 37,800$$

The smallest natural number divisible by 135, 280, and 300 is 37,800.

The least common multiple of a group of numbers can also be found by dividing by prime factors. The process is slightly different than that for finding the GCF.

**Finding the Least Common Multiple (Dividing by Prime Factors Method)**

1. Write the numbers in a row.
2. Divide each of the numbers by a common prime factor. Try 2, then try 3, and so on.
3. Divide the quotients by a common prime factor. When no prime will divide all quotients, but a prime will divide some of them, divide where possible and bring any nondivisible quotients down. Continue until no prime will divide any two quotients.
4. The product of all prime divisors from Steps 2 and 3 as well as all remaining quotients is the least common multiple.

**EXAMPLE 7** Find the least common multiple of 12, 18, and 60.

Proceed just as in Example 4 to obtain the following.

```
2 | 12  18  60
---|-------
3 | 6    9  30
---|-------
2 | 3    10
---|-----
```

The least common multiple of 135, 280, and 300 is 37,800. Compare with Example 6.
Now, even though no prime will divide 2, 3, and 10, the prime 2 will divide 2 and 10. Divide the 2 and the 10 and bring down the 3.

2) 12 18 60
   3)  6   9  30
   2)  2   3  10

\[2 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 5 = 180\]

The LCM of 12, 18, and 60 is 180.

It is shown in more advanced courses that the least common multiple of two numbers \(m\) and \(n\) can be obtained by dividing their product by their greatest common factor. (This method works only for two numbers, not for more than two.)

### Finding the Least Common Multiple (Formula)

The least common multiple of \(m\) and \(n\) is given by

\[
\text{LCM} = \frac{m \cdot n}{\text{Greatest common factor of } m \text{ and } n}.
\]

**EXAMPLE 8** Use the formula to find the least common multiple of 90 and 168.

In Example 5 we used the Euclidean algorithm to find that the greatest common factor of 90 and 168 is 6. Therefore, the formula gives us

\[
\text{Least common multiple of } 90 \text{ and } 168 = \frac{90 \cdot 168}{6} = 2520.
\]

**Problem Solving**

Problems that deal with questions such as “How many objects will there be in each group if each group contains the same number of objects?” and “When will two events occur at the same time?” can sometimes be solved using the ideas of greatest common factor and least common multiple.

**EXAMPLE 9** The King Theatre and the Star Theatre run movies continuously, and each starts its first feature at 1:00 P.M. If the movie shown at the King lasts 80 minutes and the movie shown at the Star lasts 2 hours, when will the two movies start again at the same time?

First, convert 2 hours to 120 minutes. The question can be restated as follows: “What is the smallest number of minutes it will take for the two movies to start at the same time again?” This is equivalent to saying, “What is the least common multiple of 80 and 120?” Using any of the methods described in this section, it can be shown that the least common multiple of 80 and 120 is 240. Therefore, it will take 240 minutes, or 240/60 = 4 hours for the movies to start again at the same time. By adding 4 hours to 1:00 P.M., we find that they will start together again at 5:00 P.M.
EXAMPLE 10  Joshua Hornsby has 450 football cards and 840 baseball cards. He wants to place them in stacks on a table so that each stack has the same number of cards, and no stack has different types of cards within it. What is the largest number of cards that he can have in each stack?

Here, we are looking for the largest number that will divide evenly into 450 and 840. This is, of course, the greatest common factor of 450 and 840. Using any of the methods described in this section, we find that

\[
\text{Greatest common factor of 450 and 840} = 30.
\]

Therefore, the largest number of cards he can have in each stack is 30. 

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