5.2 Selected Topics from Number Theory

Smith Numbers The mathematician Albert Wilansky, when phoning his brother-in-law, Mr. Smith, noticed an interesting property concerning Smith's phone number (493-7775). The number 4,937,775 is composite, and its prime factorization is $3 \times 5 \times 5 \times 65{,}837$. When the digits of the phone number are added, the result, 42, is equal to the sum of the digits in the prime factors: $3 + 5 + 5 + 6 + 5 + 8 + 3 + 7 = 42$. Wilansky termed such a number a *Smith number.* In 1985 it was proved that there are infinitely many Smith numbers, but there still are many unanswered questions about them.

In an earlier chapter we introduced figurate numbers, a topic investigated by the Pythagoreans. This group of Greek mathematicians and musicians held their meetings in secret, and were led by Pythagoras. In this section we examine some of the other special numbers that fascinated the Pythagoreans and are still studied by mathematicians today.

Perfect Numbers

Divisors of a natural number were covered in Section 5.1. The **proper divisors** of a natural number include all divisors of the number except the number itself. For example, the proper divisors of 8 are 1, 2, and 4. (8 is *not* a proper divisor of 8.)

Perfect Numbers

A natural number is said to be **perfect** if it is equal to the sum of its proper divisors.

Is 8 perfect? No, since $1 + 2 + 4 = 7$, and $7 \neq 8$. The smallest perfect number is 6, since the proper divisors of 6 are 1, 2, and 3, and

$$1 + 2 + 3 = 6.$$ 6 is perfect.

EXAMPLE 1 Show that 28 is a perfect number.

The proper divisors of 28 are 1, 2, 4, 7, and 14. The sum of these is 28:

$$1 + 2 + 4 + 7 + 14 = 28.$$

By the definition, 28 is perfect. ■

The numbers 6 and 28 are the two smallest perfect numbers. The next two are 496 and 8128. The pattern of these first four perfect numbers led early writers to conjecture that

1. The nth perfect number contains exactly n digits.
2. The even perfect numbers end in the digits 6 and 8, alternately.

(Exercises 41–43 will help you evaluate these conjectures.)

There still are many unanswered questions about perfect numbers. Euclid showed that if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect, and conversely. Since the prime values of $2^n - 1$ are the Mersenne primes (discussed in the previous section), this means that for every new Mersenne prime discovered, another perfect number is automatically revealed. (Hence, as of late 2002, there were also 39 known perfect numbers.) It is also known that all even perfect numbers must take the form $2^{n-1}(2^n - 1)$ and it is strongly suspected that no odd perfect numbers exist. (Any odd one would have at least eight different prime factors and would have at least 300 decimal digits.) Therefore, Euclid and the early Greeks most likely identified the form of all perfect numbers.

Deficient and Abundant Numbers

Earlier we saw that 8 is not perfect since it is greater than the sum of its proper divisors ($8 > 7$).

Weird Numbers A number is said to be *weird* if it is abundant without being equal to the sum of any set of its own proper divisors. For example, 70 is weird because it is abundant ($1 + 2 + 5 + 7 + 10 + 14 + 35 = 74 > 70$), but no set of the factors 1, 2, 5, 7, 10, 14, 35 adds up to 70.

Deficient and Abundant Numbers

A natural number is **deficient** if it is greater than the sum of its proper divisors. It is **abundant** if it is less than the sum of its proper divisors.

Based on this definition, a *deficient number does not have enough* proper divisors to add up to itself, while an *abundant number has more than enough* proper divisors to add up to itself.

EXAMPLE 2 Decide whether the given number is deficient or abundant.

(a) 12

The proper divisors of 12 are 1, 2, 3, 4, and 6. The sum of these is 16. Since $12 < 16$, the number 12 is abundant.

(b) 10

The proper divisors of 10 are 1, 2, and 5. Since $1 + 2 + 5 = 8$, and $10 > 8$, the number 10 is deficient. ■

Amicable (Friendly) Numbers

Suppose that we add the proper divisors of 284:

$$1 + 2 + 4 + 71 + 142 = 220.$$

Their sum is 220. Now, add the proper divisors of 220:

$$1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284.$$

Notice that the sum of the proper divisors of 220 is 284, while the sum of the proper divisors of 284 is 220. Number pairs such as these are said to be *amicable* or *friendly*.

Amicable or Friendly Numbers

The natural numbers a and b are **amicable,** or **friendly,** if the sum of the proper divisors of a is b, and the sum of the proper divisors of b is a.

The smallest pair of amicable numbers, 220 and 284, was known to the Pythagoreans, but it was not until more than 1000 years later that the next pair, 18,416 and 17,296, was discovered. Many more pairs were found over the next few decades, but it took a 16-year-old Italian boy named Nicolo Paganini to discover in the year 1866 that the pair of amicable numbers 1184 and 1210 had been overlooked for centuries!

Today, powerful computers continually extend the lists of known amicable pairs. The last time we checked, well over half a million pairs were known, some of them having over 5000 digits. It still is unknown, however, if there are infinitely many such pairs. Finally, no one has found an amicable pair without prime factors in common, but the possibility of such a pair has not been eliminated.

Sociable Numbers An extension of the idea of amicable numbers results in sociable numbers. In a chain of sociable numbers, the sum of the proper divisors of each number is the next number in the chain, and the sum of the proper divisors of the last number in the chain is the first number. Here is a 5-link chain of sociable numbers:

12,496
14,288
15,472
14,536
14,264.

The number 14,316 starts a 28-link chain of sociable numbers.

Goldbach's Conjecture

One of the most famous unsolved problems in mathematics is Goldbach's conjecture. The mathematician Christian Goldbach (1690–1764) conjectured (guessed) that every even number greater than 2 can be written as the sum of two prime numbers. For example, $8 = 5 + 3$ and $10 = 5 + 5$ (or $10 = 7 + 3$). Mathematicians have tried, without success, to prove Goldbach's conjecture, though it has been verified for numbers up to 4×10^{14}.

A Dull Number? The Indian mathematician Srinivasa Ramanujan (1887–1920) developed many ideas in number theory. His friend and collaborator on occasion was G. H. Hardy, also a number theorist and professor at Cambridge University in England.

A story has been told about Ramanujan that illustrates his genius. Hardy once mentioned to Ramanujan that he had just taken a taxicab with a rather dull number: 1729. Ramanujan countered by saying that this number isn't dull at all; it is the smallest natural number that can be expressed as the sum of two cubes in two different ways:

$$1^3 + 12^3 = 1729$$

and $$9^3 + 10^3 = 1729.$$

Show that 85 can be written as the sum of two squares in two ways.

EXAMPLE 3 Write each even number as the sum of two primes.

(a) 18

$18 = 5 + 13$. Another way of writing it is $7 + 11$. Notice that $1 + 17$ is *not* valid, since by definition 1 is not a prime number.

(b) 60

$60 = 7 + 53$. Can you find other ways? Why is $3 + 57$ not valid? ■

Twin Primes

Prime numbers that differ by 2 are called **twin primes.** Some twin primes are 3 and 5, 5 and 7, 11 and 13, and so on. One unproved conjecture about prime numbers (the "twin prime conjecture") states that there are infinitely many pairs of twin primes. Like Goldbach's conjecture, this has never been proved. You may wish to verify that there are eight such pairs less than 100, using the Sieve of Eratosthenes in Table 1. As of late 2002, the largest known twin primes were $318{,}032{,}361 \cdot 2^{107{,}001} \pm 1$. Each contains 32,220 digits.

Recall from Section 5.1 that Euclid's proof of the infinitude of primes used numbers of the form $p_1 \cdot p_2 \cdot p_3 \cdots p_n + 1$, where all the p's are prime. It may seem that any such number must be prime, but that is not so. (See Exercise 80 of Section 5.1.) However, this form often does produce primes (as does the same form with the plus replaced by a minus). When *all* the primes up to p_n are included, the resulting primes are called **primorial primes.** They are denoted $p\# \pm 1$. For example, $5\# + 1 = 2 \cdot 3 \cdot 5 + 1 = 31$ is a primorial prime. (In late 2002, the largest known primorial prime was $392{,}113\# + 1$, a number with 169,966 digits.) The primorial primes are a popular place to look for twin primes.

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Sophie Germain (1776–1831) studied at the École Polytechnique in Paris in a day when female students were not admitted. She submitted her work to the professor Joseph Louis Lagrange (profiled in a margin note in the Metric System appendix) under the pseudonym M. Leblanc. Lagrange eventually learned her true identity, and praised her as a promising young mathematician. Later in her career, she proved Fermat's last theorem for a special case involving exponents divisible by certain kinds of primes, which were therefore named in her honor. A **Sophie Germain prime** is an odd prime p for which $2p + 1$ also is prime. Lately, large Sophie Germain primes have been discovered at the rate of several per year. As of early 2003, the largest one known was $2{,}540{,}041{,}185 \cdot 2^{114{,}729} - 1$, which has 34,547 digits. *Source:* http://www.utm.edu/research/primes

EXAMPLE 4 Verify that the primorial formula $p\# \pm 1$ produces twin prime pairs for both **(a)** $p = 3$ and **(b)** $p = 5$.

(a) $3\# \pm 1 = 2 \cdot 3 \pm 1 = 6 \pm 1 = 5$ and 7 (twin primes)

(b) $5\# \pm 1 = 2 \cdot 3 \cdot 5 \pm 1 = 30 \pm 1 = 29$ and 31 (twin primes) ■

Fermat's Last Theorem

In any right triangle with shorter sides a and b, and longest side (hypotenuse) c, the equation $a^2 + b^2 = c^2$ will hold true. This is the famous Pythagorean Theorem. For example, with $a = 3$, $b = 4$, and $c = 5$, we have

$$3^2 + 4^2 = 5^2$$
$$25 = 25.$$

There are infinitely many such triples of numbers that satisfy this equation. Are there any triples of natural numbers that satisfy $a^n + b^n = c^n$ for natural numbers $n \geq 3$? Pierre de Fermat, profiled in a margin note in the next section, thought not and wrote in the margin of a book that he had "a truly wonderful proof" for this, but that the margin was "too small to contain it." Did he indeed have a proof, or did he have an incorrect proof?

Whatever the case, Fermat's assertion was the object of more than 400 years of attempts by mathematicians to provide a suitable proof. While it was verified for many specific cases (Fermat himself proved it for $n = 3$), a proof of the general case could not be found until the Princeton mathematician Andrew Wiles announced a proof in the spring of 1993. Although some flaws were discovered in his argument, Wiles was able, by the fall of 1994, to repair and even improve the proof.

There were probably about 100 mathematicians around the world qualified to understand the Wiles proof. Many of these examined and approved it. Some portions of it were simplified and improved. Today Fermat's Last Theorem finally is regarded by the mathematics community as officially proved.

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Andrew Wiles proved Fermat's Last Theorem

EXAMPLE 5 One of the theorems proved by Fermat is as follows: *Every odd prime can be expressed as the difference of two squares in one and only one way.* For each of the following odd primes, express it as the difference of two squares.

(a) 3

$$3 = 4 - 1 = 2^2 - 1^2$$

(b) 7

$$7 = 16 - 9 = 4^2 - 3^2$$ ■

FOR FURTHER THOUGHT

One of the most remarkable books on number theory is *The Penguin Dictionary of Curious and Interesting Numbers* (1986) by David Wells. This book contains fascinating numbers and their properties, some of which are given here.

- There are only three sets of three digits which form prime numbers in all possible arrangements: {1, 1, 3}, {1, 9, 9}, and {3, 3, 7}.
- Find the sum of the cubes of the digits of 136: $1^3 + 3^3 + 6^3 = 244$. Repeat the process with the digits of 244: $2^3 + 4^3 + 4^3 = 136$. We're back to where we started.
- 635,318,657 is the smallest number that can be expressed as the sum of two fourth powers in two ways:

$$635{,}318{,}657 = 59^4 + 158^4 = 133^4 + 134^4.$$

- The number 24,678,050 has an interesting property:

$$24{,}678{,}050 = 2^8 + 4^8 + 6^8 + 7^8 + 8^8 + 0^8 + 5^8 + 0^8.$$

- The number 54,748 has a similar interesting property:

$$54{,}748 = 5^5 + 4^5 + 7^5 + 4^5 + 8^5.$$

- The number 3435 has this property:

$$3435 = 3^3 + 4^4 + 3^3 + 5^5.$$

For anyone whose curiosity is piqued by such facts, this book is for you!

For Group Discussion

Have each student in the class choose a three-digit number that is a multiple of 3. Add the cubes of the digits. Repeat the process until the same number is obtained over and over. Then, have the students compare their results. What is curious and interesting about this process?