Earlier we introduced and studied the concept of a set, a collection of elements. A set, in itself, may have no particular structure. But when we introduce ways of combining the elements (called operations) and ways of comparing the elements (called relations), we obtain a **mathematical system**.

### Mathematical System

A **mathematical system** is made up of three components:

1. a set of elements;
2. one or more operations for combining the elements;
3. one or more relations for comparing the elements.

A familiar example of a mathematical system is the set of whole numbers \(\{0, 1, 2, 3, \ldots\}\), along with the operation of addition and the relation of equality.

Historically, the earliest mathematical system to be developed involved the set of counting numbers or initially a limited subset of the “smaller” counting numbers. The various ways of symbolizing and working with the counting numbers are called **numeration systems**. The symbols of a numeration system are called **numerals**.

### Historical Numeration Systems

Primitive societies have little need for large numbers. Even today, the languages of some cultures contain no words for numbers beyond “one,” “two,” and maybe an indefinite word suggesting “many.” For example, according to UCLA physiologist Jared Diamond (*Discover*, Aug. 1987, p. 38), there are Gimi villages in New Guinea that use just two root words—*iya* for one and *rarido* for two. Slightly larger numbers are indicated using combinations of these two: for example, *rarido-rarido* is four and *rarido-rarido-iya* is five.

A practical method of keeping accounts by matching may have developed as humans established permanent settlements and began to grow crops and raise livestock. People might have kept track of the number of sheep in a flock by matching pebbles with the sheep, for example. The pebbles could then be kept as a record of the number of sheep.

A more efficient method is to keep a **tally stick**. With a tally stick, one notch or tally is made on a stick for each sheep. Tally sticks and tally marks have been found that appear to be many thousands of years old. Tally marks are still used today: for example, nine items are tallied by writing \(\mid\mid\mid\mid\mid\mid\mid\mid\). Tally sticks and groups of pebbles were an important advance in counting. By these methods, the idea of **number** began to develop. Early people began to see that a group of three chickens and a group of three dogs had something in common: the idea of **three**. Gradually, people began to think of numbers separately from the things they represented. Words and symbols were developed for various numbers.

The numerical records of ancient people give us some idea of their daily lives and create a picture of them as producers and consumers. For example, Mary and Joseph went to Bethlehem to be counted in a census—a numerical record. Even earlier than that, as long as 5000 years ago, the Egyptian and Sumerian peoples were using large numbers in their government and business records. Ancient documents reveal some of their numerical methods, as well as those of the Greeks, Romans, Chinese, and Hindus. Numeration systems became more sophisticated as the need arose.
Ancient Egyptian Numeration—Simple Grouping Early matching and tallying led to the essential ingredient of all more advanced numeration systems, that of grouping. Grouping allows for less repetition of symbols and also makes numerals easier to interpret. Most historical systems, including our own, have used groups of ten, indicating that people commonly learn to count by using their fingers. The size of the groupings (again, usually ten) is called the base of the number system. Bases of five, twenty, and sixty have also been used.

The ancient Egyptian system is an example of a simple grouping system. It utilized ten as its base, and its various symbols are shown in Table 1. The symbol for 1 (l) is repeated, in a tally scheme, for 2, 3, and so on up to 9. A new symbol is introduced for 10 (∩), and that symbol is repeated for 20, 30, and so on, up to 90. This pattern enabled the Egyptians to express numbers up to 9,999,999 with just the seven symbols shown in the table.

<table>
<thead>
<tr>
<th>Number</th>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>l</td>
<td>Stroke</td>
</tr>
<tr>
<td>10</td>
<td>∩</td>
<td>Heel bone</td>
</tr>
<tr>
<td>100</td>
<td></td>
<td>Scroll</td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td>Lotus flower</td>
</tr>
<tr>
<td>10,000</td>
<td></td>
<td>Pointing finger</td>
</tr>
<tr>
<td>100,000</td>
<td></td>
<td>Burbot fish</td>
</tr>
<tr>
<td>1,000,000</td>
<td></td>
<td>Astonished person</td>
</tr>
</tbody>
</table>

The Egyptian symbols used denote the various powers of the base (ten):

\[
10^0 = 1, \quad 10^1 = 10, \quad 10^2 = 100, \quad 10^3 = 1000, \quad 10^4 = 10,000, \\
10^5 = 100,000, \quad \text{and} \quad 10^6 = 1,000,000.
\]

The smaller numerals at the right of the 10s, and slightly raised, are called exponents.

EXAMPLE 1 Write in our system the number below.

Refer to Table 1 for the values of the Egyptian symbols. Each \(\text{意味} \) represents 100,000. Therefore, two \(\text{意味} \) represent \(2 \times 100,000\), or 200,000. Proceed as follows:

- two \(\text{意味} \) \(2 \times 100,000 = 200,000\)
- five \(\text{意味} \) \(5 \times 1000 = 5000\)
- four \(\text{意味} \) \(4 \times 100 = 400\)
- nine \(\text{意味} \) \(9 \times 10 = 90\)
- seven \(\text{意味} \) \(7 \times 1 = 7\)

The number is 205,497.
EXAMPLE 2 Write 376,248 in Egyptian symbols.

Writing this number requires three $\text{D}_s$, seven $\text{P}_s$, six $\text{W}_s$, two $\text{S}_s$, four $\text{T}_s$, and eight $\text{L}_s$, or

\[
\text{DPPSWSTLLL}.
\]

Notice that the position or order of the symbols makes no difference in a simple grouping system. Each of the numbers $\text{LLLL}$, $\text{PPP}$, and $\text{SSSS}$ would be interpreted as 234. The most common order, however, is that shown in Examples 1 and 2, where like symbols are grouped together and groups of higher-valued symbols are positioned to the left.

A simple grouping system is well suited to addition and subtraction. For example, to add \text{DPPSWSTLLL} and \text{PPP} in the early Egyptian system, work as shown. Two $\text{L}_s$ plus six $\text{L}_s$ equal to eight $\text{L}_s$, and so on.

\[
\begin{array}{c}
\text{DPPSWSTLLL} \\
+ \text{PPP} \\
\hline
\text{DPPSWSTLLLL}
\end{array}
\]

While we used a $+$ sign for convenience and drew a line under the numbers, the Egyptians did not do this.

Sometimes regrouping, or “carrying,” is needed as in the example below in which the answer contains more than nine heel bones. To regroup, get rid of ten heel bones from the tens group. Compensate for this by placing an extra scroll in the hundreds group.

\[
\begin{array}{c}
\text{DPPSWSTLLLL} \\
+ \text{PPP} \\
\hline
\text{DPPSWSTLLLL}
\end{array}
\]

Subtraction is done in much the same way, as shown in the next example.

EXAMPLE 3 Subtract in each of the following.

\[
\begin{array}{c}
\begin{array}{c}
\text{(a)} \quad \text{999} \text{NN} \\
\text{99} \text{NN} \\
\hline
\text{999} \text{NN} \\
\text{99} \text{NN}
\end{array} \\
\text{Difference:} \quad \text{99} \text{NN}
\end{array}
\]

In part (b), to subtract four $\text{L}_s$ from two $\text{L}_s$, “borrow” one heel bone, which is equivalent to ten $\text{L}_s$. Finish the problem after writing ten additional $\text{L}_s$ on the right.
A procedure such as those described above is called an **algorithm**: a rule or method for working a problem. The Egyptians used an interesting algorithm for multiplication that requires only an ability to add and to double numbers, as shown in Example 4. For convenience, this example uses our symbols rather than theirs.

**EXAMPLE 4** A stone used in building a pyramid has a rectangular base measuring 5 by 18 cubits. Find the area of the base.

The area of a rectangle is found by multiplying the length and the width; in this problem, we must find $5 \times 18$. To begin, build two columns of numbers, as shown below. Start the first column with 1, and the second column with 18. Each column is built downward by doubling the number above. Keep going until the first column contains numbers that can be added to equal 5. Here $1 + 4 = 5$. To find $5 \times 18$, add only those numbers from the second column that correspond to 1 and 4. Here 18 and 72 are added to get the answer 90. The area of the base of the stone is 90 square cubits.

Finally, $5 \times 18 = 90$.

**EXAMPLE 5** Use the Egyptian multiplication algorithm to find $19 \times 70$.

Form two columns, headed by 1 and by 70. Keep doubling until there are numbers in the first column that add up to 19. (Here, $1 + 2 + 16 = 19$.) Then add corresponding numbers from the second column: $70 + 140 + 1120 = 1330$, so that $19 \times 70 = 1330$.

**Traditional Chinese Numeration—Multiplicative Grouping** Examples 1 through 3 above show that simple grouping, although an improvement over tallying, still requires considerable repetition of symbols. To denote 90, for example, the ancient Egyptian system must utilize nine $\text{n}$: $\text{n} \text{n} \text{n} \text{n}$. If an additional symbol (a “multiplier”) was introduced to represent nine, say “9,” then 90 could be denoted $9 \text{n}$. All possible numbers of repetitions of powers of the base could be handled by introducing a separate multiplier symbol for each counting number less than the base. Although the ancient Egyptian system apparently did not evolve in this direction, just such a system was developed many years ago in China.
It was later adopted, for the most part, by the Japanese, with several versions occurring over the years. Here we show the predominant Chinese version, which used the symbols shown in Table 2. We call this type of system a multiplicative grouping system. In general, such a system would involve pairs of symbols, each pair containing a multiplier (with some counting number value less than the base) and then a power of the base. The Chinese numerals are read from top to bottom rather than from left to right.

Three features distinguish this system from a strictly pure multiplicative grouping system. First, the number of 1s is indicated using a single symbol rather than a pair. In effect, the multiplier (1, 2, 3, ..., 9) is written but the power of the base (10^n) is not. Second, in the pair indicating 10s, if the multiplier is 1, then that multiplier is omitted. Just the symbol for 10 is written. Third, when a given power of the base is totally missing in a particular number, this omission is shown by the inclusion of the special zero symbol. (See Table 2.) If two or more consecutive powers are missing, just one zero symbol serves to note the total omission. The omission of 1s and 10s, and any other powers occurring at the extreme bottom of a numeral, need not be noted with a zero symbol. (Note that, for clarification in the examples that follow, we have emphasized the grouping into pairs by spacing and by using braces. These features are not part of the actual numeral.)

**EXAMPLE 6** Interpret the Chinese numerals below.

(a) \[ \begin{array}{l} 3 \times 1000 = 3000 \\ 1 \times 100 = 100 \\ 6 \times 10 = 60 \\ 4(\times 1) = 4 \end{array} \] Total: \( 3000 + 100 + 60 + 4 = 3164 \)

(b) \[ \begin{array}{l} 7 \times 100 = 700 \\ 0(\times 10) = 00 \\ 3(\times 1) = 3 \end{array} \] Total: \( 700 + 3 = 703 \)

(c) \[ \begin{array}{l} 5 \times 1000 = 5000 \\ 0(\times 100) = 000 \\ 0(\times 10) = 00 \end{array} \] \( 9(\times 1) = 9 \) Total: \( 5000 + 0 + 0 + 9 = 5009 \)

**EXAMPLE 7** Write Chinese numerals for these numbers.

(a) 614

This number is made up of six 100s, one 10, and one 4, as depicted at the right.

\[ \begin{array}{l} 6 \times 100: 6\,\text{百} \\ (1 \times 10): \, + \\ 4(\times 1): 4 \end{array} \]
Babylonian numeration was positional, base sixty. But the face values within the positions were base ten simple grouping numerals, formed with the two symbols shown above. (These symbols resulted from the Babylonian method of writing on clay with a wedge-shaped stylus.) The numeral

\[
\begin{align*}
\text{denotes} & \quad 1421 \ (23 \times 60 \ + \ 41 \times 1),
\end{align*}
\]

\[
\begin{align*}
\text{denotes} & \quad 5090 \quad \text{(b)}
\end{align*}
\]

The number consists of five 1000s, no 100s, and nine 10s (no 1s).

5 \times 1000: \quad \{\text{symbol}\}

0(\times \ 100): \quad \{\text{symbol}\}

9 \times 10: \quad \{\text{symbol}\}

\textbf{Hindu-Arabic Numeration—Positional System} A simple grouping system relies on repetition of symbols to denote the number of each power of the base. A multiplicative grouping system uses multipliers in place of repetition, which is more efficient. But the ultimate in efficiency is attained only when we proceed to the next step, a \textit{positional} system, in which only the multipliers are used. The various powers of the base require no separate symbols, since the power associated with each multiplier can be understood by the position that the multiplier occupies in the numeral. If the Chinese system had evolved into a positional system, then the numeral for 7482 could be written

\[
\begin{align*}
\text{rather than}
\end{align*}
\]

The lowest symbol is understood to represent two 1s \((1^0)\), the next pair up denotes eight 10s \((10^1)\), then four 100s \((10^2)\), and finally seven 1000s \((10^3)\). Each symbol in a numeral now has both a \textit{face value}, associated with that particular symbol (the multiplier value), and a \textit{place value} (a power of the base), associated with the place, or position, occupied by the symbol.

\textbf{Finger Reckoning} There is much evidence that early humans (in various cultures) used their fingers to represent numbers. As calculations became more complicated, \textit{finger reckoning}, as shown in this sketch, became popular. The Romans became adept at this sort of calculating, carrying it to 10,000 or perhaps higher.

\textbf{Positional Numeration} In a positional numeral, each symbol (called a \textit{digit}) conveys two things:

1. \textbf{face value}—the inherent value of the symbol
2. \textbf{place value}—the power of the base which is associated with the position that the digit occupies in the numeral.

The place values in a Hindu-Arabic numeral, from right to left, are 1, 10, 100, 1000, and so on. The three 4s in the number 46,424 all have the same face value but different place values. The first 4, on the left, denotes four 10,000s, the next one denotes four 100s, and the one on the right denotes four 1s. Place values (in base ten) are named as shown here:

\[
\begin{align*}
\text{Billions,} & \quad \text{Hundred millions,} \\
\text{Millions,} & \quad \text{Hundred thousands,} \\
\text{Thousands,} & \quad \text{Hundred,} \\
\text{Tens,} & \quad \text{Units,} \\
\text{Decimal point,}
\end{align*}
\]

This numeral is read as eight billion, three hundred twenty-one million, four hundred fifty-six thousand, seven hundred ninety-five.
To work successfully, a positional system must have a symbol for zero to serve as a placeholder in case one or more powers of the base are not needed. Because of this requirement, some early numeration systems took a long time to evolve to a positional form, or never did. Although the traditional Chinese system does utilize a zero symbol, it never did incorporate all the features of a positional system, but remained essentially a multiplicative grouping system.

The one numeration system that did achieve the maximum efficiency of positional form is our own system, commonly known for historical reasons as the Hindu-Arabic system. It was developed over many centuries. Its symbols have been traced to the Hindus of 200 B.C. They were picked up by the Arabs and eventually transmitted to Spain, where a late tenth-century version appeared like this:

\[
\begin{align*}
\text{I} & \quad \text{Z} & \quad \text{Z} & \quad \text{Y} & \quad \text{G} & \quad \text{J} & \quad \text{N} & \quad \text{O}.
\end{align*}
\]

The earliest stages of the system evolved under the influence of navigational, trade, engineering, and military requirements. And in early modern times, the advance of astronomy and other sciences led to a structure well suited to fast and accurate computation. The purely positional form that the system finally assumed was introduced to the West by Leonardo Fibonacci of Pisa (1170–1250) early in the thirteenth century. But widespread acceptance of standardized symbols and form was not achieved until the invention of printing during the fifteenth century. Since that time, no better system of numeration has been devised, and the positional base ten Hindu-Arabic system is commonly used around the world today. (In India, where it all began, standardization still is not totally achieved, as various local systems are used today.)

In the next section we shall look in more detail at the structure of the Hindu-Arabic system and some early methods and devices for doing computation.

### 4.1 EXERCISES

**Convert each Egyptian numeral to Hindu-Arabic form.**

1. \[\underline{\text{\hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill}}\]
2. \[\underline{\text{\hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill}}\]
3. \[\underline{\text{\hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill}}\]
4. \[\underline{\text{\hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill}}\]

**Convert each Hindu-Arabic numeral to Egyptian form.**

5. 23,145
6. 427
7. 8,657,000
8. 306,090